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Combinatorial aspects of the character variety of a family of one-relator groups

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ABSTRACT

Let us consider the group $G = \langle x, y \mid x^m = y^n\rangle$ with m and n nonzero integers. In this paper, we study the character variety $X(G)$ in $SL(2,\mathbb{C})$ of the group G, obtaining by elementary methods an explicit primary decomposition of the ideal corresponding to $X(G)$ in the coordinates $X = t_x$, $Y = t_y$ and $Z = t_{xy}$. As an easy consequence, a formula for computing the number of irreducible components of $X(G)$ as a function of m and n is given. Finally we provide a combinatorial description of $X(G)$ and we prove that in most cases it is possible to recover (m,n) from the combinatorial structure of $X(G)$.

0. Introduction

Given a finitely generated group G, the set $R(G)$ of its representations over $SL(2,\mathbb{C})$ can be endowed with the structure of an affine algebraic variety (see [12]), the same holds for the set $X(G)$ of characters of representations over $SL(2,\mathbb{C})$ (see [1]). Since different presentations of a group G give rise to isomorphic representation and character varieties; the study of geometric invariants of $R(G)$ and $X(G)$ like the dimension or the number of irreducible components is of interest in combinatorial group theory (see [10,11,16] for instance). The varieties of representations and characters have also many applications in 3-dimensional geometry and topology as can be seen in [4,7,18] for instance. In [5] and [6] some aspects of the character variety of 2-bridge knots and links were studied, note that for a 2-bridge knot (or link) the fundamental group of its complement in S^3 admits a presentation with two generators and only one relation. In [14] a geometrical description of the character variety of the torus knots $K_{m,2}$ was given.

In [2, Theor. 3.2] an explicit set of polynomials defining the character variety of a finitely presented group was given. Nevertheless this family of polynomials is not always satisfactory in order to give a geometrical description of the character variety. In this work, using elementary algebraic and arithmetic methods, we give an explicit primary decomposition of the ideal corresponding to the character variety of the group $G_{m,n} = \langle x, y \mid x^m = y^n\rangle$ with $m, n \neq 0$, thus obtaining an easy geometrical description of it. This easier description allows us to compute geometrical invariants such as the number of irreducible components and to provide a combinatorial description of $X(G)$. Also observe that if $\gcd(m,n) = 1$ then $G_{m,n}$ is precisely the fundamental group of the exterior of the (m,n)-torus knot $K_{m,n}$, thus we have obtained the character variety for any torus knot.

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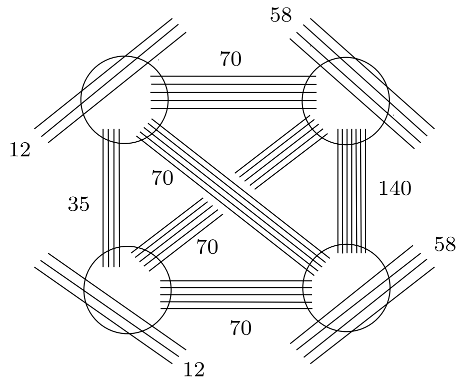

Fig. 1. Combinatorial structure of $X(G_{42,30})$.

The paper is organized as follows. In Section 1, we recall the construction of the character variety of a finitely presented group. In Section 2, we introduce some families of polynomials and give some technical results about them which are very useful in subsequent sections. Section 3 is devoted to give a complete description of the ideal associated with $X(G)$. In Section 4, we use this description to explicitly compute the number of irreducible components of $X(G)$, in particular we will proof the following result.

Theorem. *The number of irreducible components of $X(G_{m,n})$ is*:

$$\begin{cases} \frac{(|m|-1)(|n|-1)}{2} + \frac{d+1}{2} & \text{if } d \text{ is odd},\\ \frac{(|m|-1)(|n|-1)+1}{2} + \frac{d+2}{2} & \text{if } d \text{ is even}, \end{cases}$$

where $d = \gcd(m,n)$, the first summand corresponds to the number of straight lines in the projection of the set of irreducible representations of $G_{m,n}$ and the second one corresponds to the number of irreducible components in the projection of the set of reducible (*or abelian, or diagonal*) *representations of $G_{m,n}$.*

This theorem is refined in Section 5, where we study the combinatorial structure of $X(G)$. In Fig. 1, for instance, we have used our results to describe $X(G_{42,30})$. Finally in Section 6 we study how much information about m and n is enclosed in this combinatorial structure, showing that in most cases we can recover m and n from that combinatorial data.

1. Character variety of finitely presented groups

Let G be a group, a *representation* $\rho : G \to SL(2,\mathbb{C})$ is just a group homomorphism. We say that two representations ρ and ρ' are equivalent if there exists $P \in SL(2,\mathbb{C})$ such that $\rho'(g) = P^{-1}\rho(g)P$ for every $g \in G$. A representation ρ is *reducible* if the elements of $\rho(G)$ all share a common eigenvector, otherwise we say ρ is *irreducible*. The following proposition presents some useful characterizations of reducibility.

Proposition 1.1. *([1, Lem. 1.2.1 and Prop. 1.5.5])*

(1) *Let $\rho : G \to SL(2,\mathbb{C})$ be a representation. The following conditions are equivalent*:
  (a) *ρ is reducible.*
  (b) *$\rho(G)$ is, up to conjugation, a subgroup of upper triangular matrices.*
  (c) $\operatorname{tr}\rho(g) = 2$ *for all g in the commutator $G' = [G,G]$.*
(2) *If G is generated by two elements g and h, then $\rho : G \to SL(2,\mathbb{C})$ is reducible if and only if* $\operatorname{tr}\rho([g,h]) = 2$.

Now, let us consider a finitely presented group $G = \langle x_1,\dots,x_k \mid r_1,\dots,r_s\rangle$ and let $\rho : G \to SL(2,\mathbb{C})$ be a representation. It is clear that ρ is completely determined by the k-tuple $(\rho(x_1),\dots,\rho(x_k))$ and thus we can identify

$$R(G) = \big\{\big(\rho(x_1),\dots,\rho(x_k)\big) \bigm| \rho \text{ is a representation of } G\big\} \subseteq \mathbb{C}^{4k}$$

with the set of all representations of G into $SL(2,\mathbb{C})$, which is therefore (see [1]) a well-defined affine algebraic set, up to canonical isomorphism.

Recall that given a representation $\rho : G \to SL(2,\mathbb{C})$ its character $\chi_\rho : G \to \mathbb{C}$ is defined by $\chi_\rho(g) = \operatorname{tr}\rho(g)$. Note that two equivalent representations ρ and ρ' have the same character, and the converse is also true if ρ or ρ' is irreducible [1, Prop. 1.5.2]. Now choose any $g \in G$ and define $t_g : R(G) \to \mathbb{C}$ by $t_g(\rho) = \chi_\rho(g)$. It is easily seen that the ring T

generated by $\{t_g \mid g \in G\}$ is a finitely generated ring [1, Prop. 1.4.1] and, moreover, it can be shown using the well-known identities

$$\operatorname{tr} A = \operatorname{tr} A^{-1},$$
$$\operatorname{tr} AB = \operatorname{tr} BA,$$
$$\operatorname{tr} AB = \operatorname{tr} A \operatorname{tr} B - \operatorname{tr} AB^{-1} \tag{1}$$

which hold in $SL(2,\mathbb{C})$ (see [2, Cor. 4.1.2]) that T is generated by the set:

$$\{t_{x_i}, t_{x_i x_j}, t_{x_i x_j x_h} \mid 1 \leqslant i < j < h \leqslant k\}.$$

Now choose $\gamma_1, \ldots, \gamma_\nu \in G$ such that $T = \langle t_{\gamma_i} \mid 1 \leqslant i \leqslant \nu\rangle$ and define the map $t : R(G) \to \mathbb{C}^\nu$ by $t(\rho) = (t_{\gamma_1}(\rho), \ldots, t_{\gamma_\nu}(\rho))$. Observe that $\nu \leqslant \frac{k(k^2+5)}{6}$. Put $X(G) = t(R(G))$, then $X(G)$ is an algebraic variety which is well defined up to canonical isomorphism [1, Cor. 1.4.5] and is called the *character variety* of the group G in $SL(2,\mathbb{C})$. Note that $X(G)$ can be identified with the set of all characters χ_ρ of representations $\rho \in R(G)$.

For every $0 \leqslant j \leqslant k$ and for every $1 \leqslant i \leqslant s$ we have that $p_{ij} = t_{r_i x_j} - t_{x_j}$ is a polynomial with rational coefficients in the variables $\{t_{x_{i_1} \ldots x_{i_m}} \mid m \leqslant 3\}$ (see [2, Cor. 4.1.2]). Then, we have the following explicit description of $X(G)$.

Theorem 1.2. *([2, Theor. 3.2]) $X(G) = \{\bar{x} \in X(F_k) \mid p_{ij}(\bar{x}) = 0,\ \forall i, j\}$, where F_k is the free group in k generators.*

Example 1.3.

(1) $X(F_1) = \mathbb{C}$, $X(F_2) = \mathbb{C}^3$.
(2) Let $G = \langle x, y \mid xyx^{-1}y^{-1}\rangle$. It can be seen using the formulas given in (1) (see [8, Ex. 2] for instance) that $X(G) = \{(X,Y,Z) \in \mathbb{C}^3 \mid X^2 + Y^2 + Z^2 - XYZ - 4 = 0\}$. Observe that G is the fundamental group of the two-dimensional torus. In what follows we will denote $D(X,Y,Z) = X^2 + Y^2 + Z^2 - XYZ - 4$, this polynomial will play a very important role in our paper since it satisfies $D(\operatorname{tr} A, \operatorname{tr} B, \operatorname{tr} AB) = \operatorname{tr}[A,B] - 2$ for all $A, B \in SL(2,\mathbb{C})$.

Remark 1.4. Since the character variety of $X(F_1)$ is the whole field $\mathbb{C}$, the map $\operatorname{tr} : SL(2,\mathbb{C}) \to \mathbb{C}$ given by the trace of a matrix is surjective. Therefore if two polynomials in one variable coincide on $\operatorname{tr}(SL(2,\mathbb{C}))$ then they are equal as polynomials.

In the same way, since $X(F_2) = \mathbb{C}^3$, the map $t : SL(2,\mathbb{C}) \times SL(2,\mathbb{C}) \to \mathbb{C}^3$ given by $t(A,B) = (\operatorname{tr} A, \operatorname{tr} B, \operatorname{tr} AB)$ is surjective and thus if one wants to see that two polynomials in three variables are equal, it is enough to check that these polynomials coincide on $t(SL(2,\mathbb{C}) \times SL(2,\mathbb{C}))$.

Note that, since $X(F_k) \neq \mathbb{C}^{\frac{k(k^2+5)}{6}}$ for all $k \geqslant 3$, the previous considerations do not generalize for an arbitrary k. In the sequel we will make use of these observations without an explicit reference.

2. Some families of polynomials

In the forthcoming sections, we will need some particular families of polynomials whose definition and properties are given below.

Given $c_0(T), c_1(T) \in \mathbb{C}[T]$ two polynomials, we define $\mathcal{F}_k^{(c_0,c_1)}(T)$ as follows:

$$\mathcal{F}_k^{(c_0,c_1)}\colon\quad \begin{cases} \mathcal{F}_0^{(c_0,c_1)}(T) = c_0, \\ \mathcal{F}_1^{(c_0,c_1)}(T) = c_1, \\ \mathcal{F}_k^{(c_0,c_1)}(T) = T\mathcal{F}_{k-1}^{(c_0,c_1)}(T) - \mathcal{F}_{k-2}^{(c_0,c_1)}(T). \end{cases}$$

Then we denote by f_k and h_k the polynomials $\mathcal{F}_k^{(2,T)}$ and $\mathcal{F}_k^{(0,1)}$ respectively. Note that these families are closely related to the Chebyshev polynomials (see [15]), in fact it can be shown that $f_k(2X) = 2T_k(X)$ and $h_k(2X) = U_{k-1}(X)$ for all k, where T_k (resp. U_k) is the Chebyshev polynomial of the first (resp. second) kind.

Using (1) it is possible to prove that f_k is the only polynomial in one variable which satisfies $f_k(\operatorname{tr} A) = \operatorname{tr} A^k$ for all $A \in SL(2,\mathbb{C})$. Analogously, given $a, b \in \mathbb{Z}$, it can be noted that there exists only one polynomial in three variables verifying $H(\operatorname{tr} A, \operatorname{tr} B, \operatorname{tr} AB) = \operatorname{tr} A^a B^{-b}$ for all $A, B \in SL(2,\mathbb{C})$. We shall denote this polynomial by $F_{a,b}(X,Y,Z) \in \mathbb{C}[X,Y,Z]$. Now let us consider,

$$s_k(T)\colon\quad \begin{cases} s_0(T) = 0, \\ s_1(T) = s_2(T) = 1, \\ s_3(T) = T + 1, \\ s_k(T) = Ts_{k-2}(T) - s_{k-4}(T), \end{cases} \qquad \sigma_k(T)\colon\quad \begin{cases} \sigma_0(T) = 0, \\ \sigma_1(T) = \sigma_2(T) = 1, \\ \sigma_3(T) = T - 1, \\ \sigma_k(T) = T\sigma_{k-2}(T) - \sigma_{k-4}(T). \end{cases}$$

Although we have defined these families of polynomials for $k \in \mathbb{N}$, they can clearly be extended to arbitrary $k \in \mathbb{Z}$. Finally, let $\kappa : \mathbb{C}^3 \to \mathbb{C}^3$ be the involution given by $\kappa(X,Y,Z) = (-X,-Y,Z)$.

Proposition 2.1. *Let a, b, i, j, k be integers, then*:

(1) $F_{a,b}(X,Y,Z) = F_{b,a}(Y,X,Z)$, $F_{-a,-b}(X,Y,Z) = F_{a,b}(X,Y,Z)$.
(2) $F_{k,0}(X,Y,Z) = f_k(X)$, $F_{0,k}(X,Y,Z) = f_k(Y)$.
(3) $s_{-k}(T) = -s_k(T)$, $\sigma_{-k}(T) = (-1)^{k-1}\sigma_k(T)$, $h_{-k}(T) = -h_k(T)$, $f_{-k}(T) = f_k(T)$.
(4) $\kappa(f_k(X)) = (-1)^k f_k(X)$, $\kappa(s_k(X)) = (-1)^{[\frac{k-1}{2}]}\sigma_k(X)$ *and* $\kappa(F_{a,b}(X,Y,Z)) = (-1)^{a-b}F_{a,b}(X,Y,Z)$.
(5) *If m is a positive integer, then*

$$s_m(T) = \begin{cases} 1 + \sum_{i=1}^{\frac{m-1}{2}} f_i(T) & \text{if } m \text{ is odd}, \\ \sum_{i=1}^{\frac{m}{4}} f_{2i-1}(T) & \text{if } m \equiv 0, \\ 1 + \sum_{i=1}^{\frac{m-2}{4}} f_{2i}(T) & \text{if } m \equiv 2, \end{cases} \qquad (-1)^{\frac{m-1}{2}}\sigma_m(T) = 1 + \sum_{i=1}^{\frac{m-1}{2}} (-1)^i f_i(T) \quad \text{if } m \text{ is odd},$$

$$h_m(T) = \begin{cases} 1 + \sum_{i=1}^{\frac{m-1}{2}} f_{2i}(T) & \text{if } m \text{ is odd}, \\ \sum_{i=1}^{\frac{m}{2}} f_{2i-1}(T) & \text{if } m \text{ is even}, \end{cases}$$

where the congruences are taken modulo 4.
(6) $f_i(T) \cdot f_j(T) = f_{i+j}(T) + f_{i-j}(T)$.
(7) $h_{2k+1}(T) = s_{2k+1}(T)\sigma_{2k+1}(T)$, $h_{2k}(T) = s_{2k}(T) f_k(T)$.

Proof. (1)–(5) Follow by definition and/or inductive arguments. For (6) take $T = \operatorname{tr} A$ with $A \in SL(2,\mathbb{C})$. Then $f_{i+j}(T) + f_{i-j}(T) = \operatorname{tr} A^i A^j + \operatorname{tr} A^i A^{-j} = \operatorname{tr} A^i \cdot \operatorname{tr} A^j = f_i(T) \cdot f_j(T)$. (7) It is an easy consequence of (5) and (6) when k is positive. The negative case follows immediately from the positive one together with (3) and the first part of (4). □

3. Towards a primary decomposition

In what follows m and n will be assumed to be nonzero integers. Let $G_{m,n}$ be the group with presentation $G_{m,n} = \langle x, y \mid x^m = y^n \rangle$. Note that if $\gcd(m,n) = 1$ this group is isomorphic to the fundamental group of the exterior of the (m,n)-torus knot. We are interested in computing $X(G_{m,n})$, its character variety in $SL(2,\mathbb{C})$. Let $w = x^m y^{-n}$, then from Theorem 1.2, the ideal J corresponding to $X(G_{m,n})$ can be generated by $t_w - 2$, $t_{wx} - t_x$ and $t_{wy} - t_y$ in the ring of polynomials $\mathbb{C}[t_x, t_y, t_{xy}]$. In other words $J = \langle F_{m,n}(X,Y,Z) - 2, F_{m+1,n}(X,Y,Z) - X, F_{m,n-1}(X,Y,Z) - Y \rangle$, where $X = t_x$, $Y = t_y$ and $Z = t_{xy}$. Since $x^i y^{-k} = x^j y^{-l} \in G_{m,n}$ whenever $m = i - j$ and $n = k - l$, all the polynomials $t_{x^i y^{-k}} - t_{x^j y^{-l}} = F_{i,k}(X,Y,Z) - F_{j,l}(X,Y,Z)$ must belong to J. Therefore $J = \langle F_{i,k}(X,Y,Z) - F_{j,l}(X,Y,Z) \mid m = i - j,\ n = k - l \rangle \subset \mathbb{C}[X,Y,Z]$. It is also possible to verify that $F_{i,k} - F_{j,l} \in J$ when $m = i + j$ and $n = k + l$, and hence we can write, if necessary, $m = i + j$ and $n = k + l$ instead of $m = i - j$ and $n = k - l$ in the above expression of J. Now, associated with (m,n), let us define some ideals in the ring $\mathbb{C}[X,Y,Z]$ which are closely related to J.

$$I_1 = \langle s_m(X), s_n(Y) \rangle, \qquad I_2 = \begin{cases} \langle \sigma_m(X), \sigma_n(Y) \rangle & \text{if } m, n \text{ are odd}, \\ \langle \sigma_m(X), f_{\frac{n}{2}}(Y) \rangle & \text{if } m \text{ is odd and } n \text{ is even}, \\ \langle f_{\frac{m}{2}}(X), \sigma_n(Y) \rangle & \text{if } m \text{ is even and } n \text{ is odd}, \\ \langle f_{\frac{m}{2}}(X), f_{\frac{n}{2}}(Y) \rangle & \text{if } m, n \text{ are even}, \end{cases}$$
$$I_3 = J + \langle D \rangle,$$

where $D(X,Y,Z) = X^2 + Y^2 + Z^2 - XYZ - 4$ is the polynomial defined in Example 1.3(2). The aim of this section is to present the following theorem:

Theorem 3.1. $V(J) = V(I_1 \cap I_2 \cap I_3) = V(I_1) \cup V(I_2) \cup V(I_3)$.

The proof of this theorem is rather technical and will be split into two parts. In the first part we will prove that J is contained in $I_1 \cap I_2 \cap I_3$ and in the second one we will prove the inclusion $I_1 I_2 I_3 \subseteq J$. This suffices since we will have $I_1 I_2 I_3 \subseteq J \subseteq I_1 \cap I_2 \cap I_3$, consequently $\sqrt{I_1 I_2 I_3} \subseteq \sqrt{J} \subseteq \sqrt{I_1 \cap I_2 \cap I_3}$ and, as we are working over an algebraically closed field, it follows that $\sqrt{I_1 I_2 I_3} = \sqrt{I_1 \cap I_2 \cap I_3}$ which completes the proof.

3.1. The first inclusion

In order to prove that J is contained in $I_1 \cap I_2 \cap I_3$, we need some preliminary lemmas which give us some relations between $F_{a,b}$, f_k, s_k and σ_k.

Lemma 3.2. *For all integers a, b, i, j, k, l, the following expressions hold*:

(1) $F_{a,k}(X,Y,Z)f_b(X) = F_{a+b,k}(X,Y,Z) + F_{a-b,k}(X,Y,Z)$.
(2) $F_{j,a}(X,Y,Z)f_b(Y) = F_{j,a+b}(X,Y,Z) + F_{j,a-b}(X,Y,Z)$.
(3) *In particular, if $m = i - j$ is even then $F_{i,k}(X,Y,Z) + F_{j,k}(X,Y,Z) \in \langle f_{\frac{m}{2}}(X)\rangle$, and if $n = k - l$ is even then $F_{j,k}(X,Y,Z) + F_{j,l}(X,Y,Z) \in \langle f_{\frac{n}{2}}(Y)\rangle$.*

Proof. (1) Let $A, B \in SL(2,\mathbb{C})$ and put $(X,Y,Z) = (\operatorname{tr} A, \operatorname{tr} B, \operatorname{tr} AB)$. Then

$$F_{a+b,k}(X,Y,Z) + F_{a-b,k}(X,Y,Z) = \operatorname{tr} A^{a+b}B^{-k} + \operatorname{tr} A^{a-b}B^{-k} = \operatorname{tr} A^{b}A^{a}B^{-k} + \operatorname{tr} A^{-b}A^{a}B^{-k}$$
$$= \operatorname{tr} A^{b} \cdot \operatorname{tr} A^{a}B^{-k} = f_b(x)F_{a,k}(X,Y,Z).$$

(2) Follows from (1), since $F_{a,b}(X,Y,Z) = F_{b,a}(Y,X,Z)$.

(3) Take $i = a + b$ and $j = a - b$ in (1). One obtains

$$F_{i,k}(X,Y,Z) + F_{j,k}(X,Y,Z) = F_{\frac{i+j}{2},k}(X,Y,Z) f_{\frac{m}{2}}(X) \in \left\langle f_{\frac{m}{2}}(X)\right\rangle.$$

In a similar way we can prove the second part of (3). In this case, take $k = a + b$ and $l = a - b$ in (2). □

The following lemma will be useful in the sequel. Its proof is elementary and we omit.

Lemma 3.3. *Let $\{p_k\}_{k\in\mathbb{Z}}$ be a family of polynomials satisfying the recursive equation $p_k = Tp_{k-1} - p_{k-2}$ for all $k \in \mathbb{Z}$ and let $\lambda, \mu : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be two maps verifying the following conditions*:

$$\begin{cases} \lambda(i,j) = \lambda(i-1, j+1), \\ \lambda(j+1, j) = j+1, \\ \lambda(j+2, j) = j+2, \end{cases} \qquad \begin{cases} \mu(i,j) = \mu(i-1, j+1), \\ \mu(j+1, j) = j, \\ \mu(j+2, j) = j, \end{cases} \qquad \forall i, j \in \mathbb{Z}. \tag{2}$$

Then, $p_i - p_j = s_{i-j}(T)(p_{\lambda(i,j)} - p_{\mu(i,j)})$ for all $i, j \in \mathbb{Z}$.

In fact, λ and μ are uniquely determined by conditions (2).

Corollary 3.4. *Using the notation in Section* 2, *we have*:

(1) $F_{i,k}(X,Y,Z) - F_{j,k}(X,Y,Z) = s_{i-j}(X)(F_{[\frac{i+j+2}{2}],k}(X,Y,Z) - F_{[\frac{i+j-1}{2}],k}(X,Y,Z))$.
(2) $F_{i,k}(X,Y,Z) - F_{i,l}(X,Y,Z) = s_{k-l}(Y)(F_{i,[\frac{k+l+2}{2}]}(X,Y,Z) - F_{i,[\frac{k+l-1}{2}]}(X,Y,Z))$.

In particular, if $m = i - j$ then $F_{i,k}(X,Y,Z) - F_{j,k}(X,Y,Z) \in \langle s_m(X)\rangle$, and if $n = k - l$ then $F_{i,k}(X,Y,Z) - F_{i,l}(X,Y,Z) \in \langle s_n(Y)\rangle$. Moreover, if $m = i - j$ and $n = k - l$ are odd then $F_{i,k}(X,Y,Z) + F_{j,k}(X,Y,Z)$ belongs to the ideal generated by $\sigma_m(X)$ while $F_{j,k} + F_{j,l}$ belongs to the ideal generated by $\sigma_n(Y)$.

Proof. (1) Note that $\lambda(i,j) = [\frac{i+j+2}{2}]$ and $\mu(i,j) = [\frac{i+j-1}{2}]$ satisfy the conditions (2) in Lemma 3.3, and we can take $p_i(X) = F_{i,k}(X,Y,Z)$ which verifies the corresponding recursive equation. Since $F_{a,b}(X,Y,Z) = F_{b,a}(Y,X,Z)$, (2) follows from (1).

Now, assume that m and n are odd and let us apply to expression (1) the involution $\kappa : \mathbb{C}^3 \to \mathbb{C}^3$ given by $\kappa(X,Y,Z) = (-X,-Y,Z)$. Then the claim follows by Proposition 2.1 (4) and (1). □

Theorem 3.5. $J \subseteq I_1 \cap I_2 \cap I_3$.

Proof. Let i, j, k, l be integers such that $m = i - j$ and $n = k - l$. Recall that I_3 is by definition the ideal $J + \langle D\rangle$. Therefore J is contained in I_3. We divide the rest of the proof in two parts.

(1) $J \subset I_1$. The difference $F_{i,k}(X,Y,Z) - F_{j,l}(X,Y,Z)$ can be written in the form

$$F_{i,k} - F_{j,l} = (F_{i,k} - F_{j,k}) + (F_{j,k} - F_{j,l}),$$

which belongs to $\langle s_m(X), s_n(Y)\rangle = I_1$ from Corollary 3.4.

(2) $J \subset I_2$. Essentially, three cases can occur.
- m and n are odd. In this case, the involution κ can be used. Since $J \subseteq I_1$, $\kappa(J) \subseteq \kappa(I_1)$. Now observe that from the fourth part of Proposition 2.1, $\kappa(J) = J$ and $\kappa(I_1) = I_2$.
- m is odd and n is even. Here, $F_{i,k} - F_{j,l}$ can be written in the form

$$F_{i,k} - F_{j,l} = (F_{i,k} + F_{j,k}) - (F_{j,k} + F_{j,l})$$

which belongs to the ideal $\langle \sigma_m(X), f_{\frac{n}{2}}(Y)\rangle$ from Lemma 3.2 and Corollary 3.4.
- m and n are even. We write the difference $F_{i,k} - F_{j,l}$ as before. Now, from Lemma 3.2, it belongs to the ideal $\langle f_{\frac{m}{2}}(X), f_{\frac{n}{2}}(Y)\rangle$. □

3.2. The second inclusion

We shall now show that $I_1 I_2 I_3 \subseteq J$. In order to do this, it is enough to study if $I_1 I_2 \langle D \rangle$ is contained in J, since $I_3 = J + \langle D \rangle$. If necessary, we will write $J_{m,n}$ instead of just J. Since $F_{a,b}(X,Y,Z) = F_{b,a}(Y,X,Z)$, if a polynomial $H(X,Y,Z)$ belongs to $J_{m,n}$ then $H(Y,X,Z)$ belongs to $J_{n,m}$. This easy remark allows us to simplify the proofs.

Since $V(J) = X(G_{m,n}) = \{(\operatorname{tr} A, \operatorname{tr} B, \operatorname{tr} AB) \mid A, B \in SL(2,\mathbb{C}),\ A^m = B^n\}$, we can assume $A^m = B^n$ when we work modulo $J_{m,n}$.

Lemma 3.6. *$h_m(X)D \in J_{m,n}$ for all $m, n \in \mathbb{Z}$. As a consequence, $h_n(Y)D \in J_{m,n}$ for all $m, n \in \mathbb{Z}$.*

Proof. Let A, B be two matrices in $SL(2,\mathbb{C})$ and put $(X,Y,Z) = (\operatorname{tr} A, \operatorname{tr} B, \operatorname{tr} AB)$. It can easily be proved by induction on m that $h_m(X)D = \operatorname{tr} A^m B A^{-1} B^{-1} - f_{m-1}(X)$, since both members of the equality satisfy the same recursive equation. Now, recall that $A^m = B^n$ modulo $J_{m,n}$. Therefore $\operatorname{tr} A^m B A^{-1} B^{-1} \equiv \operatorname{tr} B^n A^{-1}$ and thus $h_m(X)D \equiv F_{1,n} - F_{m-1,0} \in J_{m,n}$. □

Lemma 3.7. *Take $(X,Y,Z) \in \mathbb{C}^3$ and let $A, B \in SL(2,\mathbb{C})$ be two matrices such that $(X,Y,Z) = (\operatorname{tr} A, \operatorname{tr} B, \operatorname{tr} AB)$. Then*

$$s_m(X)D = \begin{cases} \operatorname{tr} A^{\frac{m+1}{2}} B A^{-1} B^{-1} + \operatorname{tr} A^{\frac{m-1}{2}} B A^{-1} B^{-1} - \operatorname{tr} A^{\frac{m-3}{2}} - \operatorname{tr} A^{\frac{m-1}{2}} & \text{if } m \text{ is odd}, \\ \operatorname{tr} A^{\frac{m}{2}} B A^{-1} B^{-1} - \operatorname{tr} A^{\frac{m-2}{2}} & \text{if } m \text{ is even}. \end{cases}$$

Proof. As before, the claim follows since both members of the expression verify the same recursive equation and it is obvious for $m = -1, 0, 1, 2$. □

Lemma 3.8. *$s_m(X) f_{\frac{n}{2}}(Y) D \in J_{m,n}$ for all $m, n \in \mathbb{Z}$ with n even. As a consequence, $f_{\frac{m}{2}}(X) s_n(Y) D \in J_{m,n}$ for all $m, n \in \mathbb{Z}$ with m even.*

Proof. We will make use of the previous lemma, so it is clear that we must work out separately the cases m odd and m even. We will only develop the odd case here. Let us recall that $f_{\frac{n}{2}}(Y) = \operatorname{tr} B^{\frac{n}{2}}$ and apply Lemma 3.7 together with formulas (1).

$$\begin{aligned}
f_{\frac{n}{2}}(Y) s_m(X) D &= \operatorname{tr} B^{\frac{n}{2}} \big(\operatorname{tr} A^{\frac{m+1}{2}} B A^{-1} B^{-1} + \operatorname{tr} A^{\frac{m-1}{2}} B A^{-1} B^{-1} - \operatorname{tr} A^{\frac{m-3}{2}} - \operatorname{tr} A^{\frac{m-1}{2}} \big) \\
&= \operatorname{tr} B^{\frac{n-2}{2}} A^{\frac{m+1}{2}} B A^{-1} + \operatorname{tr} A^{\frac{-m-1}{2}} B^{\frac{n-2}{2}} A B^{-1} + \operatorname{tr} B^{\frac{n-2}{2}} A^{\frac{m-1}{2}} B A^{-1} \\
&\quad + \operatorname{tr} A^{\frac{-m+1}{2}} B^{\frac{n+2}{2}} A B^{-1} - \operatorname{tr} B^{\frac{n}{2}} A^{\frac{m-3}{2}} - \operatorname{tr} B^{\frac{n}{2}} A^{\frac{-m+3}{2}} - \operatorname{tr} B^{\frac{n}{2}} A^{\frac{m-1}{2}} - \operatorname{tr} B^{\frac{n}{2}} A^{\frac{-m+1}{2}} \\
&= \operatorname{tr} B^{\frac{n}{2}} \operatorname{tr} A^{\frac{m-1}{2}} B - \operatorname{tr} A^{\frac{-m-1}{2}} B^{\frac{n-2}{2}} A B^{-1} + \operatorname{tr} A^{\frac{-m-1}{2}} B^{\frac{n+2}{2}} A B^{-1} \\
&\quad + \operatorname{tr} B^{\frac{n}{2}} \operatorname{tr} A^{\frac{m-3}{2}} B - \operatorname{tr} A^{\frac{-m+1}{2}} B^{\frac{n-2}{2}} A B^{-1} + \operatorname{tr} A^{\frac{-m+1}{2}} B^{\frac{n+2}{2}} A B^{-1} \\
&\quad - \operatorname{tr} B^{\frac{n}{2}} A^{\frac{m-3}{2}} - \operatorname{tr} B^{\frac{n}{2}} A^{\frac{-m+3}{2}} - \operatorname{tr} B^{\frac{n}{2}} A^{\frac{m-1}{2}} - \operatorname{tr} B^{\frac{n}{2}} A^{\frac{-m+1}{2}} \\
&= \operatorname{tr} B^{\frac{n-4}{2}} A^{\frac{-m+1}{2}} - \operatorname{tr} A^{\frac{-m-1}{2}} B^{\frac{n-2}{2}} \operatorname{tr} A B^{-1} + \operatorname{tr} A^{\frac{-m-3}{2}} B^{\frac{n}{2}} + \operatorname{tr} A^{\frac{-m-1}{2}} B^{\frac{n+2}{2}} \operatorname{tr} A B^{-1} \\
&\quad - \operatorname{tr} A^{\frac{-m-3}{2}} B^{\frac{n+4}{2}} + \operatorname{tr} B^{\frac{n-4}{2}} A^{\frac{-m+3}{2}} - \operatorname{tr} A^{\frac{-m+1}{2}} B^{\frac{n-2}{2}} \operatorname{tr} A B^{-1} + \operatorname{tr} A^{\frac{-m-1}{2}} B^{\frac{n}{2}} \\
&\quad + \operatorname{tr} A^{\frac{-m+1}{2}} B^{\frac{n+2}{2}} \operatorname{tr} A B^{-1} - \operatorname{tr} A^{\frac{-m-1}{2}} B^{\frac{n+4}{2}} - \operatorname{tr} B^{\frac{n}{2}} A^{\frac{-m+3}{2}} - \operatorname{tr} B^{\frac{n}{2}} A^{\frac{-m+1}{2}} \\
&= \big(\operatorname{tr} A^{\frac{-m-3}{2}} B^{\frac{n}{2}} - \operatorname{tr} A^{\frac{-m+3}{2}} B^{\frac{n}{2}} \big) + \big(\operatorname{tr} A^{\frac{-m-1}{2}} B^{\frac{n}{2}} - \operatorname{tr} A^{\frac{-m+1}{2}} B^{\frac{n}{2}} \big) \\
&\quad + \big(\operatorname{tr} A^{\frac{-m+1}{2}} B^{\frac{n-4}{2}} - \operatorname{tr} A^{\frac{-m-1}{2}} B^{\frac{n+4}{2}} \big) + \big(\operatorname{tr} A^{\frac{-m+3}{2}} B^{\frac{n-4}{2}} - \operatorname{tr} A^{\frac{-m-3}{2}} B^{\frac{n+4}{2}} \big) \\
&\quad + \operatorname{tr} A B^{-1} \Big[\big(\operatorname{tr} A^{\frac{-m-1}{2}} B^{\frac{n+2}{2}} - \operatorname{tr} A^{\frac{-m+1}{2}} B^{\frac{n-2}{2}} \big) + \big(\operatorname{tr} A^{\frac{-m+1}{2}} B^{\frac{n+2}{2}} - \operatorname{tr} A^{\frac{-m-1}{2}} B^{\frac{n-2}{2}} \big) \Big] \\
&= \big(F_{\frac{m+3}{2},\frac{n}{2}} - F_{\frac{m-3}{2},\frac{n}{2}} \big) + \big(F_{\frac{m+1}{2},\frac{n}{2}} - F_{\frac{m-1}{2},\frac{n}{2}} \big) + \big(F_{\frac{m-1}{2},\frac{n-4}{2}} - F_{\frac{m+1}{2},\frac{n+4}{2}} \big) \\
&\quad + \big(F_{\frac{m-3}{2},\frac{n-4}{2}} - F_{\frac{m+3}{2},\frac{n+4}{2}} \big) + F_{1,1} \Big[\big(F_{\frac{m+1}{2},\frac{n+2}{2}} - F_{\frac{m-1}{2},\frac{n-2}{2}} \big) + \big(F_{\frac{m-1}{2},\frac{n+2}{2}} - F_{\frac{m+1}{2},\frac{n-2}{2}} \big) \Big].
\end{aligned} \tag{3}$$

Thus $s_m(X) f_{\frac{n}{2}}(Y) D \in J_{m,n}$ as claimed. □

Lemma 3.9. *$s_m(X)\sigma_n(Y)D \in J_{m,n}$ for all $m, n \in \mathbb{Z}$ with n odd. As a consequence, $\sigma_m(X) s_n(Y) D \in J_{m,n}$ for all $m, n \in \mathbb{Z}$ with m odd.*

Proof. We will carry out the case when m is odd and n is positive (the case when n is negative follows from the positive one and from Proposition 2.1 (3) and (1)). First of all note that, n being odd, $(-1)^{\frac{n-1}{2}} \sigma_n(Y) = 1 + \sum_{j=1}^{\frac{n-1}{2}} (-1)^j \operatorname{tr} B^j$. Now applying again Lemma 3.7 we can show, after some straightforward computations as in the previous lemma, that:

$$s_m(X)\sigma_n(Y)D = (F_{\frac{m+3}{2},\frac{n-1}{2}} - F_{\frac{m-3}{2},\frac{n+1}{2}}) + (F_{\frac{m+1}{2},\frac{n-1}{2}} - F_{\frac{m-1}{2},\frac{n+1}{2}}) + (F_{\frac{m-1}{2},\frac{n+3}{2}} - F_{\frac{m+1}{2},\frac{n-3}{2}}) + (F_{\frac{m-3}{2},\frac{n+3}{2}} - F_{\frac{m+3}{2},\frac{n-3}{2}}) + F_{1,-1}\big[(F_{\frac{m+1}{2},\frac{n-1}{2}} - F_{\frac{m-1}{2},\frac{n+1}{2}}) + (F_{\frac{m-1}{2},\frac{n-1}{2}} - F_{\frac{m+1}{2},\frac{n+1}{2}})\big] \tag{4}$$

and thus $s_m(X)\sigma_n(Y)D \in J_{m,n}$ as claimed. □

The last two lemmas can also be proved by double induction on m and n, since both members of the corresponding expression satisfy the same recursive equation and hence the proofs are reduced to the base cases. However, we did not proceed in that direction because finding those identities would require to work out all the computations as in the proof of Lemma 3.8. We refer to the appendix of the preprint [13] for complete details on how to find and show those complicated formulas.

Theorem 3.10. $I_1 I_2 I_3 \subseteq J$.

Proof. The above two lemmas imply that $I_1 I_2\langle D\rangle \subseteq J$. Now since $I_3 = J + \langle D\rangle$, we have that $I_1 I_2 I_3 = I_1 I_2(J + \langle D\rangle) = I_1 I_2 J + I_1 I_2\langle D\rangle \subseteq J$. □

As we already remarked, Theorems 3.5 and 3.10 obviously imply Theorem 3.1. Based on computational evidence, the authors conjecture that in fact $J = I_1 \cap I_2 \cap I_3$, but unfortunately no proof has been found.

4. Irreducible components of $X(G_{m,n})$

From Theorem 3.1, the character variety $X(G_{m,n})$ can be decomposed into the algebraic sets $V(I_1)$, $V(I_2)$ and $V(I_3)$. Therefore, in order to obtain an explicit description of $X(G_{m,n})$, we need to factorize the polynomials $s_k(T)$, $\sigma_k(T)$ and $f_k(T)$, and to find a nicer expression for $I_3 = J + \langle D\rangle$.

4.1. Finding the factorization of $s_k(T)$, $\sigma_k(T)$ and $f_k(T)$

In this section k will be a nonzero integer. Let us recall the recursive definition of the cyclotomic polynomials $\{c_\ell(T)\}_{\ell\in\mathbb{N}}$ given by

$$\prod_{\ell\mid k} c_\ell(T) = T^k - 1, \tag{5}$$

where $c_\ell(T)$ is the minimal polynomial of the primitive ℓ-th roots of unity. In the same way we will denote by $r_\ell(T)$ the minimal polynomial of the ℓ-th primitive roots of -1. This definition gives an expression similar to (5):

$$\prod_{\substack{\ell\mid k\\ \frac{k}{\ell}\text{ odd}}} r_\ell(T) = T^k + 1. \tag{6}$$

In fact it can be seen that if ℓ is odd $r_\ell(T) = c_\ell(-T)$ and if ℓ is even $r_\ell(T) = c_{2\ell}(T)$.

Note that for every $3 \leqslant \ell \in \mathbb{N}$, there exists a polynomial (irreducible over $\mathbb{Z}$) $q_\ell(T)$ such that:

$$c_\ell(T) = T^{\frac{\varphi(\ell)}{2}} q_\ell\left(T + \frac{1}{T}\right),$$

where φ is Euler's phi function and $\varphi(\ell) = \deg c_\ell(T)$. For the sake of completeness, we will denote $q_1(T) = T - 2$ and $q_2(T) = T + 2$.

Lemma 4.1. *If $\ell \geqslant 3$, $q_\ell(T)$ has $\frac{\varphi(\ell)}{2}$ distinct real roots and its set of zeroes is $Z[q_\ell] = \{2\,\mathrm{Re}\,z \mid z \text{ is a primitive } \ell\text{-th root of unity}\}$ for all $\ell \in \mathbb{N}$. Moreover if $\ell_1 \neq \ell_2$, then $Z[q_{\ell_1}] \cap Z[q_{\ell_2}] = \emptyset$.*

We can now obtain a factorization of the polynomial f_k which will allow us to find its roots.

Proposition 4.2. $f_k(T) = \prod_{\ell\mid k,\ \frac{k}{\ell}\text{ odd}} q_{4\ell}(T)$.

Proof.

$$f_k\left(T+\frac{1}{T}\right)=T^k+\frac{1}{T^k}=\frac{T^{2k}+1}{T^k}=\frac{1}{T^k}\prod_{\substack{\ell|2k\\ \frac{2k}{\ell}\ \text{odd}}}r_\ell(T)=\frac{1}{T^k}\prod_{\substack{\ell|2k\\ \frac{2k}{\ell}\ \text{odd}}}T^{\varphi(\ell)}q_{2\ell}\left(T+\frac{1}{T}\right)$$

$$=\prod_{\substack{\ell|2k\\ \frac{2k}{\ell}\ \text{odd}}}q_{2\ell}\left(T+\frac{1}{T}\right)=\prod_{\substack{\ell|k\\ \frac{k}{\ell}\ \text{odd}}}q_{4\ell}\left(T+\frac{1}{T}\right),$$

where the last equality follows readily from the fact that $\{2\ell \mid \ell|2k,\ \frac{2k}{\ell}\ \text{odd}\}=\{4\ell \mid \ell|k,\ \frac{k}{\ell}\ \text{odd}\}$ and from the identity $\sum_{\ell|2k,\ \frac{2k}{\ell}\ \text{odd}}\varphi(\ell)=k$. □

Corollary 4.3. *The polynomial $f_k(T)$ has k distinct real roots and its set of zeroes is $Z[f_k]=\{2\operatorname{Re} z \mid z$ is a primitive 4ℓ-th root of unity, $\frac{k}{\ell}$ odd$\}=\{2\operatorname{Re} z \mid z^{2k}=-1\}$.*

Proof. The first equality is an immediate consequence of the previous proposition and Lemma 4.1. The second one can be verified by direct calculations. □

Now, we turn to the polynomials s_k and σ_k. First we present a technical lemma that can easily be proved by induction.

Lemma 4.4.

$$s_k\left(T+\frac{1}{T}\right)=\begin{cases}\frac{T^k-1}{T^{\frac{k-1}{2}}(T-1)} & \text{if } k \text{ is odd},\\ \frac{T^k-1}{T^{\frac{k-2}{2}}(T^2-1)} & \text{if } k \text{ is even}.\end{cases}$$

Using this result together with (5) and Lemma 4.1, and recalling Proposition 2.1, we have the following result.

Proposition 4.5. $s_k(T)=\prod_{1,2\neq\ell|k}q_\ell(T)$, $\sigma_k(T)=(-1)^{[\frac{k-1}{2}]}\prod_{1,2\neq\ell|k}q_\ell(-T)$.

Hence, $s_k(T)$ and $\sigma_k(T)$ have $[\frac{k-1}{2}]$ distinct real roots. In particular their sets of zeroes are $Z[s_k]=\{2\operatorname{Re} z \mid z\neq\pm1;\ z^k=1\}=-Z[\sigma_k]$.

Remark 4.6. In so far k was implicitly assumed to be positive. Observe that if we shift the sign of k the sets of zeroes described above remain invariant. Namely $Z[f_k]=Z[f_{-k}]$, $Z[s_k]=Z[s_{-k}]$ and $Z[\sigma_k]=Z[\sigma_{-k}]$. Moreover, if k is negative, then $f_k(T)$ has $|k|$ distinct real roots while $s_k(T)$ and $\sigma_k(T)$ have $[\frac{|k|-1}{2}]$.

As a consequence of the previous discussion we will be able to count the number of irreducible components of $V(I_1)$ and $V(I_2)$ which, due to the form of the ideals I_1 and I_2, are disjoint straight lines.

4.2. Another description for $V(I_3)$

Now, we are interested in computing the number of irreducible components of the affine algebraic variety associated with I_3. For this aim, our definition of $I_3=J+\langle D\rangle$ doesn't seem to be very useful. That is why we need another expression for $\mathcal{C}:=V(I_3)$. From now on we assume that $d=\gcd(m,n)$ and we will write $m=m'd$ and $n=n'd$.

Lemma 4.7. $V(I_3)=\{(u+u^{-1},v+v^{-1},uv+(uv)^{-1}) \mid u,v\in\mathbb{C}^*,\ u^m=v^n\}$.

Proof. Let $t:R(G_{m,n})\to X(G_{m,n})$ be the polynomial map defined in Section 1 and let us recall that by definition $V(J)=X(G_{m,n})=t(R(G_{m,n}))$. Since $D(t(\rho))=\operatorname{tr}\rho[x,y]-2$ when $\rho\in R(G_{m,n})$ and $V(I_3)=V(J+\langle D\rangle)=V(J)\cap V(D)$, it is clear that $V(I_3)=\{t(\rho)\mid \rho\in R(G_{m,n}),\ \operatorname{tr}\rho[x,y]=2\}$. Now the claim follows from Proposition 1.1(2) and the identity $t(\mathit{Red})=t(\mathit{Diag})$, where *Red* is the set of all reducible representations in $R(G_{m,n})$ and *Diag* the set of all diagonal ones (see the proof of Corollary 1.4.5 in [1]). □

Using the previous lemma, since $u^m-v^n=\prod_{i=0}^{d-1}(u^{m'}-\zeta^i v^{n'})$ where $\{1,\dots,\zeta^{d-1}\}$ is the set of all d-th roots of unity, the variety associated with I_3 can be decomposed as the union $\bigcup_{i=0}^{d-1}C_{\zeta^i}$ with $C_{\zeta^i}=\{(u+u^{-1},v+v^{-1},uv+(uv)^{-1})\mid u,v\in\mathbb{C}^*,$ $u^{m'}=\zeta^i v^{n'}\}$. Once we remove the redundant components, the above union provides a decomposition of $\mathcal{C}=V(I_3)$ into irreducible components. This fact is clarified in the lemma below.

Lemma 4.8. *The following properties hold*:

(1) C_{ζ^i} *is an irreducible algebraic set for all i.*
(2) $C_{\zeta^i} = C_{\zeta^j}$ *if and only if* $\zeta^i = \zeta^{\pm j}$*. Moreover, if* $C_{\zeta^i} \neq C_{\zeta^j}$ *then they are disjoint.*

Proof. (1) Let us first show that the set C_{ζ^i} is algebraic. Take $Diag \subset R(G_{m,n})$ the algebraic set of all diagonal representations and $Diag^{\zeta^i}$ the algebraic subset

$$Diag^{\zeta^i} = \left\{ \left[\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix} \right] \in SL(2,\mathbb{C})^2 \;\middle|\; u^{m'} = \zeta^i v^{n'} \right\}.$$

Since the map $\mathbb{C}^* \to \mathbb{C}$ given by $a \mapsto a + a^{-1}$ is proper with the usual topology over $\mathbb{C}$, then so is $t|_{Diag}$ and, consequently, also $t|_{Diag^{\zeta^i}}$ and it follows (see [17] and [3]) that $t|_{Diag^{\zeta^i}}$ is closed with the Zariski topology. Now observe that $C_{\zeta^i} = t(Diag^{\zeta^i})$ and $Diag^{\zeta^i}$ is Zariski-closed (see also the proof of Corollary 1.4.5 in [1]). Finally, the irreducibility of C_{ζ^i} follows from the irreducibility of $Diag^{\zeta^i}$ which is birationally equivalent to the plane curve given by the equation $u^{m'} = \zeta^i v^{n'}$.

(2) Note that the equation $a + a^{-1} = b + b^{-1}$ with $a, b \in \mathbb{C}^*$ has the only solutions $a = b^{\pm 1}$. Now, the assertion is a clear consequence of this fact. □

Using a parametrization of the plane curve given by the equation $u^{m'} = \zeta^i v^{n'}$ we can also parametrize C_{ζ^i}. In this way it is easy to compute the self-intersection and regular points of C_{ζ^i}. Since the rational morphism $\mathbb{C}^* \to \mathbb{C} : t \mapsto t + 1/t$ is a $2:1$ branched covering, it is also easy to prove that C_{ζ^i} is smooth. Finally, all of its components being smooth and disjoint, we conclude that $\mathcal{C}$ is smooth.

4.3. *Counting the irreducible components of $X(G_{m,n})$*

Finally we present the main result of this section, which allows us to explicitly count the number of irreducible components of $X(G_{m,n})$.

Theorem 4.9. *The number of irreducible components of* $X(G_{m,n})$ *is*:

$$\begin{cases} \frac{(|m|-1)(|n|-1)}{2} + \frac{d+1}{2} & \text{if } d \text{ is odd}, \\ \frac{(|m|-1)(|n|-1)+1}{2} + \frac{d+2}{2} & \text{if } d \text{ is even}, \end{cases}$$

where the first summand corresponds to the number of straight lines in $V(I_1) \cup V(I_2)$ *and the second one corresponds to the number of irreducible components of* $V(I_3)$.

Remark 4.10. Note that if $d = 1$ the genus of the torus knot $K_{m,n}$ is precisely the number of straight lines in its character variety. It would be interesting to find if this is true for other groups, such as the 2-bridge knots.

In [9] the set $\widehat{R}(G_{m,n})$ of conjugacy classes of irreducible representations of $G_{m,n}$ over $SU(2)$ was studied. In particular it was proved that $\widehat{R}(G_{m,n})$ is the disjoint union of $\frac{(|m|-1)(|n|-1)}{2}$ open arcs. Let us see how this result relates to our work. First of all, recall that $SU(2) = \{ \left(\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a} \end{smallmatrix}\right) \mid a, b \in \mathbb{C},\ |a|^2 + |b|^2 = 1\} \subset SL(2,\mathbb{C})$. Thus, it makes sense to consider $t|_{\widehat{R}(G_{m,n})}$ and we have that:

$$t\big(\widehat{R}(G_{m,n})\big) = \big(V(I_1) \cup V(I_2)\big) \cap \big\{(X,Y,Z) \in \mathbb{R}^3 \;\big|\; -2 < X, Y, Z < 2\big\}.$$

5. Combinatorial description of $X(G_{m,n})$

In this section we are interested in studying the combinatorial structure of $X(G)$ as well as in describing the information that can be obtained from it. Recall that the character varieties of G associated with (m,n), $(m,-n)$ and (n,m) are all isomorphic, so from now on we will assume that $m \geqslant n$ are positive integers.

We know that the straight lines of $\mathcal{L} := V(I_1) \cap V(I_2)$ are disjoint and so are the irreducible components of $\mathcal{C} = V(I_3)$. Therefore in order to study the combinatorial structure of $X(G) = \mathcal{L} \cup \mathcal{C}$ it is enough to study how the straight lines and the irreducible components of $\mathcal{C}$ intersect. The following result will be useful in the sequel.

Lemma 5.1. $\#L \cap \mathcal{C} = 2$ *for all* $L \in \mathcal{L}$.

Proof. Consider $L : \{x = a,\ y = b\}$; i.e., a is a root of one of the polynomials s_m, σ_m or $f_{\frac{m}{2}}$ and b is a root of one of the polynomials s_n, σ_n or $f_{\frac{n}{2}}$ depending on the parity of m and n. Now take λ and μ such that $a = \lambda + \lambda^{-1}$ and $b = \mu + \mu^{-1}$.

It is clear that $L \cap \mathcal{C}$ is contained in $\{(a, b, \lambda\mu + \lambda^{-1}\mu^{-1}), (a, b, \lambda\mu^{-1} + \lambda^{-1}\mu)\} = \{P_1, P_2\}$. We need to prove that these two points are different and they indeed belong to $\mathcal{C}$. If $P_1 = P_2$ then $\lambda\mu + \lambda^{-1}\mu^{-1} = \lambda\mu^{-1} + \lambda^{-1}\mu$ and thus $\lambda = \pm 1$ or $\mu = \pm 1$. This contradicts Lemma 4.4 and Proposition 4.2. Now, note that $P_1, P_2 \in \mathcal{C}$ if and only if $\lambda^m = \mu^n$ and $\lambda^m = \mu^{-n}$. The claim follows again from Lemma 4.4 and Proposition 4.2. □

Remark 5.2. In fact it can be shown that if a line of the form $L : \{x = a, y = b\}$ and $\mathcal{C}$ intersect exactly in two different points, then L must be in $\mathcal{L}$.

In the light of the previous lemma and given $L \in \mathcal{L}$ we see that L can only intersect $\mathcal{C}$ in one or two of its components. It is clear that a point (a, b, c) with $a = \lambda + \lambda^{-1}$ and $b = \mu + \mu^{-1}$ belongs to C_{ζ^i} if and only if $\lambda^{m'} = \zeta^{\pm i}\mu^{\pm n'}$. This means that $L : \{x = a, y = b\}$ intersects $\mathcal{C}$ only in the components $C_{\lambda^{m'}\mu^{n'}}$ and $C_{\lambda^{m'}\mu^{-n'}}$ (which can be equal or not). After this discussion and Lemma 4.8, the following result is satisfied.

Proposition 5.3. *Let $L : \{x = a, y = b\}$ be a straight line of $\mathcal{L}$ and let λ, μ be two numbers such that $a = \lambda + \lambda^{-1}$ and $b = \mu + \mu^{-1}$. Then $C_{\lambda^{m'}\mu^{n'}}$ and $C_{\lambda^{m'}\mu^{-n'}}$ are the only components of $\mathcal{C}$ which intersect L. In particular L intersects only one component of $\mathcal{C}$ at two different points if and only if $\lambda^{m'} = \pm 1$ or $\mu^{n'} = \pm 1$.*

We have just seen that a line $L \in \mathcal{L}$ intersects $\mathcal{C}$ in two different points which can lie either in the same irreducible component of $\mathcal{C}$ or in different ones. Now we will compute the number of straight lines in each situation.

5.1. Straight lines intersecting only one component

We will start by computing the number of lines of $\mathcal{L}$ intersecting $\mathcal{C}$ in only one component. Of course this component can be C_1, C_{-1} (only if d is even) and C_{ζ^i} (with $i \neq 0, \frac{d}{2}$).

Proposition 5.4. *The number of straight lines intersecting C_1 twice is $\frac{(m'-1)(n'-1)}{2}$. Moreover, the previous number can be decomposed as follows:*

$$\frac{(m'-1)(n'-1)}{2} = \begin{cases} \frac{(m'-1)(n'-1)}{4} + \frac{(m'-1)(n'-1)}{4} & \text{if } d, m', n' \text{ odd}, \\ \frac{(m'-2)(n'-1)}{4} + \frac{m'(n'-1)}{4} & \text{if } d, n' \text{ odd}, \ m' \text{ even}, \\ \frac{(m'-1)(n'-1)}{2} + 0 & \text{if } d \text{ even}, \end{cases}$$

where the first (resp. second) summand corresponds to the number of straight lines in $V(I_1)$ (resp. $V(I_2)$).

Proof. We will assume that d, m' and n' are odd. The other three possible cases follow from similar considerations (observe that m' and n' cannot be both even). Recall that $\mathcal{L} = V(I_1) \cup V(I_2)$, so we will count the lines from $V(I_1)$ and $V(I_2)$ separately.

Let $L : \{x = a, y = b\}$ be a straight line from $V(I_1)$, then $a = \lambda + \lambda^{-1}$ and $b = \mu + \mu^{-1}$ with $\lambda^m = \mu^n = 1$ and $a, b \neq \pm 2$. From Proposition 5.3 it follows that $\lambda^{m'} = \mu^{n'} = \pm 1$ and since d is odd it must be $\lambda^{m'} = \mu^{n'} = 1$. Now, the previous system of complex equations has $(m'-1)(n'-1)$ solutions in (λ, μ) (recall that $\lambda, \mu \neq \pm 1$) but we are interested in the possible values of (a, b). Clearly $(\lambda^{\pm}, \mu^{\pm})$ provide the same value (a, b) thus, the number of lines from $V(I_1)$ intersecting C_1 twice is $\frac{(m'-1)(n'-1)}{4}$.

On the other hand if $L : \{x = a, y = b\}$ is a straight line in $V(I_2)$, then $a = \lambda + \lambda^{-1}$ and $b = \mu + \mu^{-1}$ with $\lambda^m = \mu^n = -1$. Analogously to the previous paragraph, we conclude that the number of straight lines in $V(I_2)$ intersecting C_1 twice is again $\frac{(m'-1)(n'-1)}{4}$ and the proposition follows. □

Proposition 5.5. *If d is even, the number of straight lines intersecting C_{-1} twice is $\frac{(m'-1)(n'-1)}{2}$. Moreover, all such straight lines are in $V(I_1)$.*

Proof. We will only work out the case when m' is even and n' is odd, the other case being analogous. As in the previous proposition we will compute the lines coming from $V(I_1)$ and $V(I_2)$ separately.

Let $L : \{x = a, y = b\}$ be a straight line in $V(I_1)$, then $a = \lambda + \lambda^{-1}$ and $b = \mu + \mu^{-1}$ with $\lambda^m = \mu^n = 1$ and $a, b \neq \pm 2$. From Proposition 5.3 it follows that $\lambda^{m'} = -\mu^{n'} = \pm 1$. Now, the system of complex equations $\lambda^{m'} = 1 = -\mu^{n'}$ has $(m'-2)(n'-1)$ solutions in (λ, μ) while the system $\lambda^{m'} = -\mu^{n'} = -1$ has $m'(n'-1)$. Again, since we are interested in the possible values of (a, b), it follows that the number of lines from $V(I_1)$ intersecting C_{-1} twice is $\frac{(m'-1)(n'-1)}{2}$.

In this case, if $L : \{x = a, y = b\}$ is a straight line in $V(I_2)$, then $a = \lambda + \lambda^{-1}$ and $b = \mu + \mu^{-1}$ with $\lambda^m = \mu^n = -1$. If L intersects C_{-1} twice, then it follows from Proposition 5.3 that $\lambda^{m'} = -\mu^{n'} = \pm 1$. Since d is even this is a contradiction. Hence there are no lines in $V(I_2)$ intersecting C_{-1} twice and the proof is complete. □

Proposition 5.6. *If $i \neq 0, \frac{d}{2}$ the number of straight lines that intersect C_{ζ^i} twice is $(m'-1)n' + m'(n'-1)$. Moreover, the previous number can be decomposed as follows*:

$$\begin{cases} \frac{(m'-1)n'+m'(n'-1)}{2} & + & \frac{(m'-1)n'+m'(n'-1)}{2} & \text{if } d, m', n' \text{ odd}, \\ \frac{(m'-2)n'+m'(n'-1)}{2} & + & \frac{m'n'+m'(n'-1)}{2} & \text{if } d, n' \text{ odd}, \ m' \text{ even}, \\ (m'-1)n' + m'(n'-1) & + & 0 & \text{if } d \text{ even}, \end{cases}$$

where the first (*resp. second*) *summand corresponds to the number of straight lines in $V(I_1)$* (*resp. $V(I_2)$*).

Proof. We will assume that d is odd, m' is even and n' is odd; the other three cases are analogous.

Let $L : \{x = a, y = b\}$ be a straight line from $V(I_1)$, then $a = \lambda + \lambda^{-1}$ and $b = \mu + \mu^{-1}$ with $\lambda^m = \mu^n = 1$ and $a, b \neq \pm 2$. From Proposition 5.3 it follows (d being odd) that $\lambda^{m'} = 1$ or $\mu^{n'} = 1$ and that $\lambda^{m'}\mu^{n'} = \zeta^{\pm i}$. We must now consider four cases:

(1) $\lambda^{m'} = 1, \ \mu^{n'} = \zeta^i$. (3) $\lambda^{m'} = \zeta^i, \ \mu^{n'} = 1$.
(2) $\lambda^{m'} = 1, \ \mu^{n'} = \zeta^{-i}$. (4) $\lambda^{m'} = \zeta^{-i}, \ \mu^{n'} = 1$.

Since we are only interested in finding solutions for (a, b) it is clear that we must only take into account cases (1) and (3). Now, case (1) provides $\frac{m'-2}{2}n'$ solutions in (a, b) while case (3) provides $m'\frac{n'-1}{2}$. Thus, the number of straight lines in $V(I_1)$ that intersect C_{ζ^i} twice is $\frac{(m'-2)n'+m'(n'-1)}{2}$.

On the other hand, let $L : \{x = a, y = b\}$ be a straight line in $V(I_2)$. Then $a = \lambda + \lambda^{-1}$ and $b = \mu + \mu^{-1}$ with $\lambda^m = \mu^n = -1$ and $a, b \neq \pm 2$. From Proposition 5.3 it follows (d being odd) that $\lambda^{m'} = -1$ or $\mu^{n'} = -1$ and that $\lambda^{m'}\mu^{n'} = \zeta^{\pm i}$. Again we must consider four cases:

(1) $\lambda^{m'} = -1, \ \mu^{n'} = \zeta^i$. (3) $\lambda^{m'} = \zeta^i, \ \mu^{n'} = -1$.
(2) $\lambda^{m'} = -1, \ \mu^{n'} = \zeta^{-i}$. (4) $\lambda^{m'} = \zeta^{-i}, \ \mu^{n'} = -1$.

And we must only take cases (1) and (3) into account. Case (1) provides $\frac{m'}{2}n'$ solutions for (a, b) while case (3) provides $m'\frac{n'-1}{2}$. Thus, the number of straight lines in $V(I_2)$ that intersect C_{ζ^i} twice is $\frac{m'n'+m'(n'-1)}{2}$ and the proposition follows. □

5.2. Straight lines intersecting different components

Now we will compute the number of lines of $\mathcal{L}$ intersecting $\mathcal{C}$ in exactly two components. These pairs of components are $\{C_{\zeta^i}, C_{\zeta^j}\}$ with $i \neq j$. We will say that a line L intersects the pair $\{C_{\zeta^i}, C_{\zeta^j}\}$ if L intersects C_{ζ^i} in one point and C_{ζ^j} in another one.

The following elementary result will be useful later.

Lemma 5.7. *Let ν be the number of complex solutions of the system $\{\omega^2 = \zeta^i, \omega^d = \pm 1\}$ under the equivalence relation $\omega_1 \sim \omega_2 \Leftrightarrow \omega_1 = \omega_2^{-1}$. Then*

$$\nu = \begin{cases} 1 & \text{if } i = \frac{d}{2}, \\ 2 & \text{otherwise}. \end{cases}$$

We would like to compute the number of straight lines of $\mathcal{L}$ intersecting $\{C_{\zeta^i}, C_{\zeta^j}\}$ with $i \neq j$. Let $L : \{x = a, y = b\}$ be a straight line of $\mathcal{L}$ and take λ, μ such that $a = \lambda + \lambda^{-1}$ and $b = \mu + \mu^{-1}$. Then from Proposition 5.3 our problem is equivalent to finding the number of solutions in (a, b) of the "system" $\{C_{\lambda^{m'}\mu^{n'}}, C_{\lambda^{m'}\mu^{-n'}}\} = \{C_{\zeta^i}, C_{\zeta^j}\}$ with $i \neq j$. We have to consider eight cases:

(1) $\lambda^{m'}\mu^{n'} = \zeta^i, \ \lambda^{m'}\mu^{-n'} = \zeta^j$. (5) $\lambda^{m'}\mu^{n'} = \zeta^i, \ \lambda^{m'}\mu^{-n'} = \zeta^{-j}$.
(2) $\lambda^{m'}\mu^{n'} = \zeta^j, \ \lambda^{m'}\mu^{-n'} = \zeta^i$. (6) $\lambda^{m'}\mu^{n'} = \zeta^{-j}, \ \lambda^{m'}\mu^{-n'} = \zeta^i$.
(3) $\lambda^{m'}\mu^{n'} = \zeta^{-i}, \ \lambda^{m'}\mu^{-n'} = \zeta^{-j}$. (7) $\lambda^{m'}\mu^{n'} = \zeta^{-i}, \ \lambda^{m'}\mu^{-n'} = \zeta^j$.
(4) $\lambda^{m'}\mu^{n'} = \zeta^{-j}, \ \lambda^{m'}\mu^{-n'} = \zeta^{-i}$. (8) $\lambda^{m'}\mu^{n'} = \zeta^j, \ \lambda^{m'}\mu^{-n'} = \zeta^{-i}$.

Like in the proof of Proposition 5.6 we only have to take cases (1) and (5) into account, since the other ones do not provide more solutions for (a, b). Moreover, if i or j equals 0 or $\frac{d}{2}$ then only the case (1) has to be considered.

Proposition 5.8. *The number of straight lines of $\mathcal{L}$ which intersect $\{C_{\zeta^i}, C_{\zeta^j}\}$ with $i \neq j$ at two different points is*

$$\begin{cases} m'n' & \text{if } (i, j) = (0, \frac{d}{2}), \\ 2m'n' & \text{if } i \neq 0, \frac{d}{2}, \ j = 0, \\ 2m'n' & \text{if } i \neq 0, \frac{d}{2}, \ j = \frac{d}{2}, \\ 4m'n' & \text{if } i, j \neq 0, \frac{d}{2}. \end{cases}$$

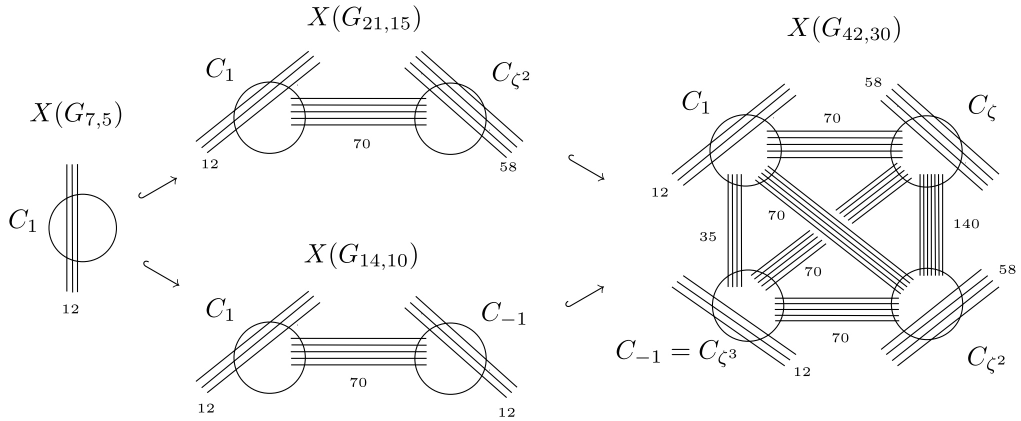

Fig. 2. Combinatorial structure of $X(G_{7,5})$, $X(G_{21,15})$, $X(G_{14,10})$ and $X(G_{42,30})$.

Proof. Let us first assume that $i \neq 0, \frac{d}{2}$ and $j = 0$. Then from the above considerations we have that $\lambda^{m'}\mu^{n'} = \zeta^i$, $\lambda^{m'}\mu^{-n'} = 1$ and thus $\lambda^{2m'} = \zeta^i$, $\mu^{n'} = \lambda^{m'}$. From Lemma 5.7 the number of complex solutions for (a,b) of the previous system is clearly $\nu m'n' = 2m'n'$.

The first and third cases are completely analogous. For the last case $i, j \neq 0, \frac{d}{2}$, $i \neq j$ we have take into account two different systems $\{\lambda^{m'}\mu^{n'} = \zeta^i, \lambda^{m'}\mu^{-n'} = \zeta^j\}$, $\{\lambda^{m'}\mu^{n'} = \zeta^i, \lambda^{m'}\mu^{-n'} = \zeta^{-j}\}$. Using again Lemma 5.7, one can see that both systems have $\nu m'n' = 2m'n'$ solutions and the proof is complete. □

5.3. Combinatorial structure of $X(G)$

Now we are ready to present the main result of this paper that allow one to describe the combinatorial structure of the character variety as a function of m and n. Recall that d is the greatest common divisor of m, n and $m = m'd$, $n = n'd$. Note that the proof is obvious after the above discussion.

Theorem 5.9. *The number of straight lines of $\mathcal{L}$ can be decomposed as follows.*

(a) *If d is odd,*

$$\frac{(m'-1)(n'-1)}{2} + \frac{d-1}{2}\cdot\big[(m'-1)n' + m'(n'-1)\big] + \frac{d-1}{2}\cdot 2m'n' + \binom{(d-1)/2}{2}\cdot 4m'n',$$

where the first (resp. second) summand corresponds to the number of straight lines intersecting C_1 (resp. C_{ζ^i} with $i \neq 0$) twice and the third (resp. fourth) one corresponds to the number of straight lines intersecting $\{C_1, C_{\zeta^i}\}$ with $i \neq 0$ (resp. $\{C_{\zeta^i}, C_{\zeta^j}\}$ with $i, j \neq 0$, $i \neq j$) at two different points.

(b) *If d is even,*

$$\frac{(m'-1)(n'-1)}{2} + \frac{(m'-1)(n'-1)}{2} + \frac{d-2}{2}\cdot\big[(m'-1)n' + m'(n'-1)\big] + m'n' + \frac{d-2}{2}\cdot 2m'n' + \frac{d-2}{2}\cdot 2m'n' + \binom{(d-2)/2}{2}\cdot 4m'n',$$

where the first (resp. second; third) summand corresponds to the number of straight lines intersecting C_1 (resp. C_{-1}; C_{ζ^i} with $i \neq 0, \frac{d}{2}$) twice and the fourth (resp. fifth; sixth; seventh) one corresponds to the number of straight lines intersecting $\{C_1, C_{-1}\}$ (resp. $\{C_1, C_{\zeta^i}\}$ with $i \neq 0, \frac{d}{2}$; $\{C_{-1}, C_{\zeta^i}\}$ with $i \neq 0, \frac{d}{2}$; $\{C_{\zeta^i}, C_{\zeta^j}\}$ with $i, j \neq 0, \frac{d}{2}$, $i \neq j$) at two different points.

Example 5.10. Let us use the previous theorem to describe the combinatorial structure of $X(G_{42,30})$. First, note that if $a|m$ and $b|n$ then there exists a surjective group homomorphism $G_{m,n} \twoheadrightarrow G_{a,b}$ induced by the identity on F_2. Now, it is easy to construct an injective polynomial map $X(G_{a,b}) \hookrightarrow X(G_{m,n})$ coming from the identity on $\mathbb{C}^3$. Thus we have the following embeddings $X(G_{7,5}) \subset X(G_{21,15}) \subset X(G_{42,30})$ and $X(G_{7,5}) \subset X(G_{14,10}) \subset X(G_{42,30})$. In Fig. 2 we show the combinatorial structure of $X(G_{42,30})$ and how $X(G_{7,5})$, $X(G_{14,10})$ and $X(G_{21,15})$ are embedded in it.

6. The variety $X(G)$ as an invariant of G

Once we have given a complete description of the combinatorial structure of $X(G)$, it naturally arises the question of how much information about m and n is codified in this structure. Clearly, if $d = 1$, it will not be possible to recover any

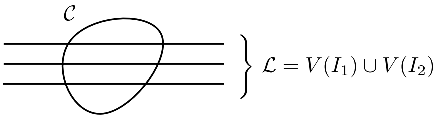

Fig. 3. Combinatorial structure of $(7,2)$ and $(4,3)$.

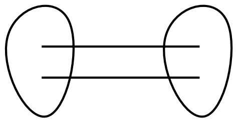

Fig. 4. Combinatorial structure of $(3,3)$ and $(4,2)$.

information from the combinatorial structure of $X(G)$, since it only consists of one irreducible component $V(I_3)$ cut by $\frac{(m-1)(n-1)}{2}$ disjoint straight lines and this latter value does not determine m and n even if they are coprime. For instance $(7,2)$ and $(4,3)$ have the same combinatorial structure (see Fig. 3). Nevertheless, for $d \neq 1$ and with only one exception, we will show that it is possible to recover the value of m and n only from the combinatorial structure of $X(G)$.

Firstly, let us suppose that every irreducible component of $X(G)$ contains exactly 2 singularities. Then, by the previous section, it is easy to see that the possible values for (m,n) are $(3,2)$, $(3,3)$, and $(4,2)$. Since the number of irreducible components of $X(G_{3,2})$ is 2, while $X(G_{3,3})$ and $X(G_{4,2})$ both of them contain 4 irreducible components, we can recognize the case $(m,n)=(3,2)$. Unfortunately the combinatorial structure of $X(G_{3,3})$ and $X(G_{4,2})$ are identical and it does not allow us to distinguish these cases (see Fig. 4).

Now, we must study the case when some of the irreducible components of $X(G)$ contain a number of singular points different from 2. These components must be precisely those coming from $V(I_3)$; let us denote them by $A_1,\dots,A_k$. Of course, if $k=1$, then $d=1$ and we have no possibility of obtaining m and n. On the other hand, if $k \geqslant 2$, let us denote by a_{ij} the number of straight lines that intersect only A_i and A_j and put $M=(a_{ij})_{1\leqslant i,j\leqslant k}$ which is a $k\times k$ symmetric matrix over the integers. Note that, for every $i\in\{1,\dots,k\}$, the sum $s_i = 2a_{ii} + \sum_{i\neq j=1}^{k} a_{ij}$ is the number of singular points of A_i and consequently $s_i \neq 2$ for all i. By the previous section, and after a permutation of rows and columns (which preserves the set of diagonal elements of M), we can suppose that

$$M=\begin{pmatrix} \frac{(m'-1)(n'-1)}{2} & 2m'n' & 2m'n' & \cdots & 2m'n' \\ & (m'-1)n'+m'(n'-1) & 4m'n' & \cdots & 4m'n' \\ & & (m'-1)n'+m'(n'-1) & \cdots & 4m'n' \\ & & & \ddots & \vdots \\ & & & & (m'-1)n'+m'(n'-1) \end{pmatrix} \quad \text{if } d \text{ is odd,}$$

$$M=\begin{pmatrix} \frac{(m'-1)(n'-1)}{2} & m'n' & 2m'n' & \cdots & 2m'n' \\ & \frac{(m'-1)(n'-1)}{2} & 2m'n' & \cdots & 2m'n' \\ & & (m'-1)n'+m'(n'-1) & \cdots & 4m'n' \\ & & & \ddots & \vdots \\ & & & & (m'-1)n'+m'(n'-1) \end{pmatrix} \quad \text{if } d \text{ is even.}$$

With these matrices in mind and recalling the previous section we can perform the following analysis:

- If $\operatorname{tr} M = 0$, then at least $n'=1$ and three cases can occur:
  (1) If $k=2$, then $a_{12}\neq 2$ and it follows that $d=2$, $m'=a_{12}$ and $(m,n)=(2a_{12},2)$.
  (2) If $k>2$ and $a_{ij}=1$ for some $i\neq j$, then $m'=n'=1$ and $d=2k-2$.
  (3) If $k>2$ and $a_{ij}\neq 1$ for every $i\neq j$ then again $m'=n'=1$, but $d=2k-1$ and we are done.
- If $\operatorname{tr} M \neq 0$, then define $a=\min\{a_{ii} \mid 1\leqslant i\leqslant k\}$. Two cases can occur:
  (1) If a appears only once in the diagonal of M, then $d=2k-1$, $a=\frac{(m'-1)(n'-1)}{2}$ and $\min\{a_{ij} \mid i\neq j\}=2m'n'$ so we obtain m and n by elementary methods.
  (2) If a appears twice in the diagonal of M, then $d=2k-2$, $a=\frac{(m'-1)(n'-1)}{2}$ and $\min\{a_{ij} \mid i\neq j\}=m'n'$ and we again recover the values of m and n.

Remark 6.1. We have seen that given the combinatorial structure of $X(G_{m,n})$ we can recover the values of m and n in most cases. Unfortunately there are situations where the combinatorial structure is not enough to obtain the values of m and n; in other words, the combinatorial structure of $X(G)$ is not a complete invariant. Nevertheless in such situation we will always have only a finite number of possibilities for the pair (m,n). This is clear in the case $(3,3)$ and $(4,2)$, but it is also true if $d=1$. We will give an example.

Let us suppose that we are given the combinatorial structure of $X(G)$ and we find that $d=1$ while the number of straight lines obtained is 18. Then we know that $(m-1)(n-1)=36$ and consequently the only possibilities (m and n being coprime) are $(37,2)$, $(19,3)$ and $(13,4)$. This situation is general by virtue of the prime decomposition. However, even if $d=1$ there are cases where we can recover m and n. For instance if there are p straight lines where p is a prime of the form $3k+2$ (there are infinitely many) the only possible solution for (m,n) is $(2p+1,2)$.

It would be interesting to find or construct another invariant from the character variety which could be shown to be complete.

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