Periods on Arithmetic Moduli Spaces

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Outline

1. Overview of the problem.
2. The $r = 1$ case.
3. The $r = 2$ case.
4. Future work.

Statement of the Problem

- We study the dynamics of the action of several monoids/groups of morphisms of F_r (e.g. injections, general automorphisms, outer automorphisms) on the character variety $Hom(F_r, SL(2, \mathbb{F}_q))//SL(2, \mathbb{F}_q)$.
- In particular, we characterize the orbits, provide criterion for determining periodic and preperiodic points, and compute the periods. We also work on visualizing the dynamics (orbits, functional graphs, etc.). We are concerned with $r \geq 1$ and $\mathbb{F}_q$ of odd order.
- We have classified when the points of $Hom(F_1, SL(2, \mathbb{F}_q))//SL(2, \mathbb{F}_q)$ are periodic and preperiodic, and we have also begun to classify when the periods of the points of $Hom(F_2, SL(2, \mathbb{F}_q))//SL(2, \mathbb{F}_q)$.

Dynamical System

Let S be a set and let $F : S \to S$ be a map from S to itself. The iterate of F with itself n times is denoted

$$F^{(n)} = F \circ F \circ \cdots \circ F$$

.

A point $P \in S$ is **periodic** if $F(n)(P) = P$ for some $n > 1$. The point is **preperiodic** if $F(k)(P)$ is periodic for some $k \geq 1$.
The (forward) orbit of P is the set

$$O_F(P) = \left\{ P, F(P), F^{(2)}(P), F^{(3)}(P), \cdots \right\}.$$

Thus P is preperiodic if and only if its orbit $O_F(P)$ is finite.

The Setup

Define $Out(F_r) := Aut(F_r)/Inn(F_r)$, where $Aut(F_r)$ and $Inn(F_r)$ are the automorphisms and inner automorphisms of the free group of rank r, respectively. Consider $Q := Hom(F_r, SL_n(\mathbb{F}_q))/SL_n(\mathbb{F}_q)$ and let $Out(F_r)$ act on Q.

The Process

1. Fix $[\alpha] \in Out(F_r)$ and $[f] \in Q$.
2. Choose $\alpha^{'} \in [\alpha]$ and $f^{'} \in [f]$.
3. Compute $\alpha(f^{'}) := f^{'} \circ \alpha^{'}$.
4. Find $[\alpha^{'}(f^{'})] \in Q$ and iterate.

This defines a dynamical system. As $\mathbb{F}_q$ is a finite field, it is reasonable to ask whether there exist periodic orbits.

How to identify $Hom(F_r, \mathrm{SL}(2, \mathbb{F}_q)), Out(F_r)$

- Identify $\phi \in Hom(F_2, \mathrm{SL}(2, \mathbb{F}_q))$ with $(\phi(a), \phi(b))$ where $F_2 = F(\{a, b\})$.
- For larger r, identify $\phi \in Hom(F_r, \mathrm{SL}(2, \mathbb{F}_q))$ with $Hom(F_r, \mathrm{SL}(2, \mathbb{F}_q))$ with $(\phi(a_1), \phi(a_2), ..., \phi(a_r))$ where $F_r = F(\{a_1, ..., a_r\})$.
- To identify $Out(F_2)$, it has been shown that the maps $\eta : (a, b) \to (ab, b)$, $\tau : (a, b) \to (b, a), \iota : (a, b) \to (a^{-1}, b)$ generate $Out(F_2)$.

F_r when r = 1

- Since F_1 is cyclic, the only self-homomorphisms are $a \to a^n$ for $n \in \mathbb{Z}$, so $Aut(F_1) = \{id, -id\}$, and $Inn(F_1)$ is the trivial group
- This suggests viewing a different class of morphisms for $r = 1$, in which case we chose the "analogue" $Onj(F_1) = Inj(F_1)/Inn(F_1)$, where $Inj(F_1)$ are the monomorphisms of F_1 to itself.
- Then for any $n \geq 1$ consider the power map $P_n : SL(2, \mathbb{F}_q) \to SL(2, \mathbb{F}_q)$ defined by $P_n([A]) = [A^n]$.

F_r where r = 1, cont'd

We had the following table for orders of elements in $\mathrm{SL}(2, \mathbb{F}_q)$

Conjugacy class type	Representative	Order
$\pm I$	$\begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}$	If I, 1; if -I, 2
Parabolic ($\alpha \in \mathbb{F}_q$, $\alpha = 1$ or $\alpha \neq \omega^2$)	$A = \begin{bmatrix} \pm 1 & \alpha * \\ 0 & \pm 1 \end{bmatrix}$	If tr(A)=2, char($\mathbb{F}_q$); if tr(A) = -2, 2char($\mathbb{F}_q$)
Diagonalizable over $\mathbb{F}_q$	$\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$	$\frac{q-1}{\gcd(\bar{a}**, q-1)}$
Diagonalizable over $\mathbb{F}_{q^2}$	$\begin{bmatrix} c & 0 \\ 0 & c^{-1} \end{bmatrix}$	$\frac{q^2-1}{\gcd(\tilde{c}***, q^2-1)}$ (will divide $q+1$)

- When r=1, we showed that a matrix was strictly periodic if and only if it had order relatively prime to the exponent n of the map $\phi : a \to a^n$.

  In this table:

- *$\alpha = 1$ or is not a square in $\mathbb{F}_q$.
- **$\bar{a}$ represents $\phi(a)$ where $\phi : (\mathbb{F}_q, \cdot) \to (\mathbb{Z}_{q-1}, +)$
- ***$\tilde{c}$ represents $\gamma(c)$ where $\gamma : (\mathbb{F}_{q^2}, \cdot) \to (\mathbb{Z}_{q^2-1}, +)$ with ϕ, γ being isomorphisms.

F_r when r = 2

- The traces of the generators A, B, AB parameterize $Hom(F_2, SL(2, \mathbb{F}_q))/SL(2, \mathbb{F}_q)$ as the affine space $\mathbb{F}_q^3$
- The character map $Tr : Hom(F_2, \mathrm{SL}(2, \mathbb{F}_q))/\mathrm{SL}(2, \mathbb{F}_q) \to \mathbb{F}_q^3$ given by
$$[[A, B]] \mapsto (trA, trB, tr(AB))$$
is an isomorphism.
- We substitute the original setting for the dynamical system with this induced action.
- For $r \geq 2$, all points are periodic. We turn to maximum orbit length to further study the conjugacy classes.

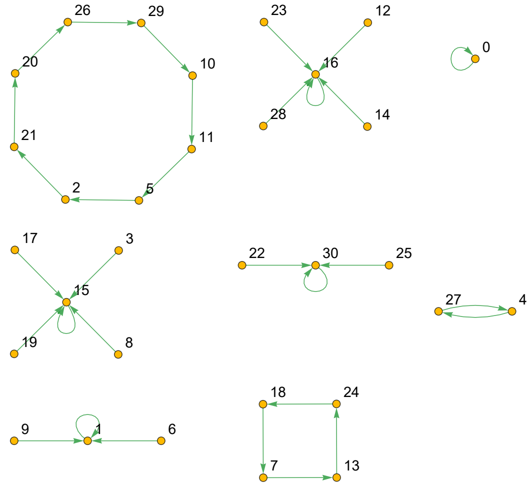
F_r when r = 2, cont'd

The action of $Out(F_2)$ on $SL(2, \mathbb{F}_q)$ induces an equivariant action on $\mathbb{F}_q^3$. We get the following table

	(A, B)	$(trA, trB, trAB)$	(x, y, z)
ι	(A^{-1}, B)	$(trA^{-1}, trB, trA^{-1}B)$	$(x, y, xy - z)$
τ	(B, A)	$(trB, trA, trBA)$	(y, x, z)
η	(AB, B)	$(trAB, trB, trAB^2)$	$(z, y, yz - x)$
η^{-1}	(AB^{-1}, B)	$(trAB^{-1}, trB, trA)$	$(xy - z, y, x)$

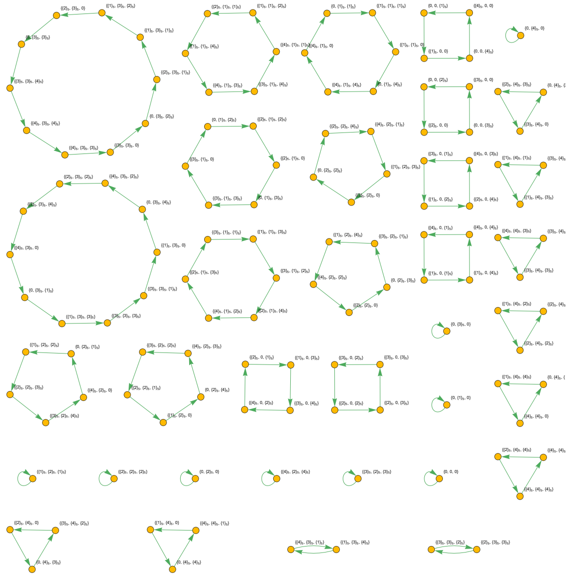
Visualization r = 1

Graph of the 5th Chebyshev Polynomial of the first type $T_5(x) = 16x^5 - 20x^3 + 5x$ acting on $\mathbb{Z}_{31}$.



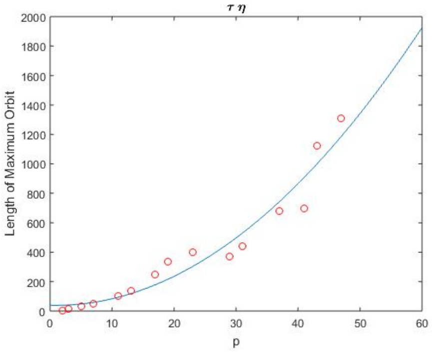
Visualization r = 2

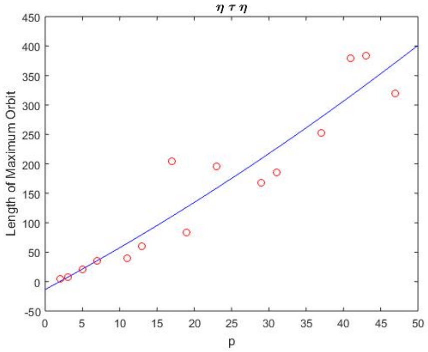
η maps $(x, y, z) \to (z, y, xy - z)$ in $\mathbb{Z}_5$



Length of Maximum Orbits

The following plots depict the length of the maximum orbit versus the prime p for a length two and length three word, respectively.





Period Data

p	L(p)		p	L(p)
2	4		2	4
3	12		3	8
5	30		5	20
7	48		7	36
11	102		11	40
13	120		13	60
17	138		17	204
19	246		19	84
23	336		23	196
29	399		29	168
31	372		31	186

Questions

- Question: How many α's in $Out(F_2)$ are needed to make $\bigcup_\alpha\{\alpha^\kappa(\mathrm{x}_{\mathrm{max}})|\kappa \geq 0\} = \kappa^{-1}(\kappa(\mathrm{x}_{max}))$?

- Find a pair of matrices that realize the dip. Explore why the form of those matrices gave us a dip in the first place.

Future Work

- Study maximum orbit lengths in order to learn a tight bound on the largest period while varying primes.
- Aim to formulate a similar study varying degree of $\mathbb{F}_q$ over $\mathbb{F}_p$ for a fixed prime p.