Periods on Arithmetic Moduli Spaces

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Acknowledgments

- Dr. Sean Lawton
- The Mason Experimental Geometry Lab
Outline

1. Overview of the problem.
2. The $r = 1$ case.
3. The $r = 2$ case.
4. Future work.
Statement of the Problem

- We study the dynamics of the action of several monoids/groups of morphisms of $F_r$ (e.g. injections, general automorphisms, outer automorphisms) on the character variety $\text{Hom}(F_r, SL(2, \mathbb{F}_q))/\!/SL(2, \mathbb{F}_q)$.

- In particular, we characterize the orbits, provide criterion for determining periodic and preperiodic points, and compute the periods. We also work on visualizing the dynamics (orbits, functional graphs, etc.). We are concerned with $r \geq 1$ and $\mathbb{F}_q$ of odd order.

- We have classified when the points of $\text{Hom}(F_1, SL(2, \mathbb{F}_q))/\!/SL(2, \mathbb{F}_q)$ are periodic and preperiodic, and we have also begun to classify when the periods of the points of $\text{Hom}(F_2, SL(2, \mathbb{F}_q))/\!/SL(2, \mathbb{F}_q)$.
Dynamical System

Let $S$ be a set and let $F : S \rightarrow S$ be a map from $S$ to itself. The iterate of $F$ with itself $n$ times is denoted

$$F^{(n)} = F \circ F \circ \cdots \circ F$$

A point $P \in S$ is periodic if $F^{(n)}(P) = P$ for some $n > 1$. The point is preperiodic if $F^{(k)}(P)$ is periodic for some $k \geq 1$. The (forward) orbit of $P$ is the set

$$O_F(P) = \left\{ P, F(P), F^{(2)}(P), F^{(3)}(P), \cdots \right\}.$$

Thus $P$ is preperiodic if and only if its orbit $O_F(P)$ is finite.
The Setup

Define $Out(F_r) := Aut(F_r)/Inn(F_r)$, where $Aut(F_r)$ and $Inn(F_r)$ are the automorphisms and inner automorphisms of the free group of rank $r$, respectively. Consider $Q := Hom(F_r, SL_n(\mathbb{F}_q))/SL_n(\mathbb{F}_q)$ and let $Out(F_r)$ act on $Q$.

The Process

1. Fix $[\alpha] \in Out(F_r)$ and $[f] \in Q$.
2. Choose $\alpha' \in [\alpha]$ and $f' \in [f]$.
3. Compute $\alpha(f') := f' \circ \alpha'$.
4. Find $[\alpha'(f')] \in Q$ and iterate.

This defines a dynamical system. As $\mathbb{F}_q$ is a finite field, it is reasonable to ask whether there exist periodic orbits.
How to identify $\text{Hom}(F_r, \text{SL}(2, \mathbb{F}_q))$, $\text{Out}(F_r)$

- Identify $\phi \in \text{Hom}(F_2, \text{SL}(2, \mathbb{F}_q))$ with $(\phi(a), \phi(b))$ where $F_2 = F(\{a, b\})$.

- For larger $r$, identify $\phi \in \text{Hom}(F_r, \text{SL}(2, \mathbb{F}_q))$ with $\text{Hom}(F_r, \text{SL}(2, \mathbb{F}_q))$ with $(\phi(a_1), \phi(a_2), \ldots, \phi(a_r))$ where $F_r = F(\{a_1, \ldots, a_r\})$.

- To identify $\text{Out}(F_2)$, it has been shown that the maps $\eta : (a, b) \rightarrow (ab, b)$, $\tau : (a, b) \rightarrow (b, a)$, $\iota : (a, b) \rightarrow (a^{-1}, b)$ generate $\text{Out}(F_2)$. 
Since $F_1$ is cyclic, the only self-homomorphisms are $a \to a^n$ for $n \in \mathbb{Z}$, so $\text{Aut}(F_1) = \{id, -id\}$, and $\text{Inn}(F_1)$ is the trivial group.

This suggests viewing a different class of morphisms for $r = 1$, in which case we chose the "analogue" $\text{Onj}(F_1) = \text{Inj}(F_1)/\text{Inn}(F_1)$, where $\text{Inj}(F_1)$ are the monomorphisms of $F_1$ to itself.

Then for any $n \geq 1$ consider the power map $P_n : \text{SL}(2, \mathbb{F}_q) \to \text{SL}(2, \mathbb{F}_q)$ defined by $P_n([A]) = [A^n]$. 
We had the following table for orders of elements in $\text{SL}(2, \mathbb{F}_q)$

<table>
<thead>
<tr>
<th>Conjugacy class type</th>
<th>Representative</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pm I$</td>
<td>$\begin{pmatrix} \pm 1 &amp; 0 \ 0 &amp; \pm 1 \end{pmatrix}$</td>
<td>$\text{If } I, 1; \text{ if } -I, 2$</td>
</tr>
<tr>
<td>Parabolic ($\alpha \in \mathbb{F}_q$, $\alpha = 1$ or $\alpha \neq \omega^2$)</td>
<td>$A = \begin{pmatrix} \pm 1 &amp; \alpha^* \ 0 &amp; \pm 1 \end{pmatrix}$</td>
<td>If $\text{tr}(A)=2$, $\text{char}(\mathbb{F}_q)$; if $\text{tr}(A) = -2$, $2\text{char}(\mathbb{F}_q)$</td>
</tr>
<tr>
<td>Diagonalizable over $\mathbb{F}_q$</td>
<td>$\begin{pmatrix} a &amp; 0 \ 0 &amp; a^{-1} \end{pmatrix}$</td>
<td>$\frac{q-1}{\gcd(\overline{a}^*, q-1)}$</td>
</tr>
<tr>
<td>Diagonalizable over $\mathbb{F}_{q^2}$</td>
<td>$\begin{pmatrix} c &amp; 0 \ 0 &amp; c^{-1} \end{pmatrix}$</td>
<td>$\frac{q^2-1}{\gcd(\tilde{c}**, q^2-1)}$ (will divide $q + 1$)</td>
</tr>
</tbody>
</table>

When $r=1$, we showed that a matrix was strictly periodic if and only if it had order relatively prime to the exponent $n$ of the map $\phi : a \rightarrow a^n$.

In this table:

- $\alpha = 1$ or is not a square in $\mathbb{F}_q$.
- $\tilde{a}$ represents $\phi(a)$ where $\phi : (\mathbb{F}_q, \cdot) \rightarrow (\mathbb{Z}_{q-1}, +)$
- $\tilde{c}$ represents $\gamma(c)$ where $\gamma : (\mathbb{F}_{q^2}, \cdot) \rightarrow (\mathbb{Z}_{q^2-1}, +)$ with $\phi, \gamma$ being isomorphisms.
Fr when r = 2

- The traces of the generators A, B, AB parameterize $\text{Hom}(F_2, \text{SL}(2, \mathbb{F}_q))/\text{SL}(2, \mathbb{F}_q)$ as the affine space $\mathbb{F}_q^3$.
- The character map $\text{Tr} : \text{Hom}(F_2, \text{SL}(2, \mathbb{F}_q))/\text{SL}(2, \mathbb{F}_q) \to \mathbb{F}_q^3$ given by

  \[
  [[A, B]] \mapsto (\text{tr}A, \text{tr}B, \text{tr}(AB))
  \]

  is an isomorphism.
- We substitute the original setting for the dynamical system with this induced action.
- For $r \geq 2$, all points are periodic. We turn to maximum orbit length to further study the conjugacy classes.
Fr when r = 2, cont’d

The action of \( Out(F_2) \) on \( SL(2, \mathbb{F}_q) \) induces an equivariant action on \( \mathbb{F}_q^3 \). We get the following table

<table>
<thead>
<tr>
<th>( \iota )</th>
<th>( (A, B) )</th>
<th>( (\text{tr}A, \text{tr}B, \text{tr}AB) )</th>
<th>( (x, y, z) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tau )</td>
<td>( (A^{-1}, B) )</td>
<td>( (\text{tr}A^{-1}, \text{tr}B, \text{tr}A^{-1}B) )</td>
<td>( (x, y, xy - z) )</td>
</tr>
<tr>
<td>( \eta )</td>
<td>( (B, A) )</td>
<td>( (\text{tr}B, \text{tr}A, \text{tr}BA) )</td>
<td>( (y, x, z) )</td>
</tr>
<tr>
<td>( \eta^{-1} )</td>
<td>( (AB, B) )</td>
<td>( (\text{tr}AB, \text{tr}B, \text{tr}AB^2) )</td>
<td>( (z, y, yz - x) )</td>
</tr>
<tr>
<td></td>
<td>( (AB^{-1}, B) )</td>
<td>( (\text{tr}AB^{-1}, \text{tr}B, \text{tr}A) )</td>
<td>( (xy - z, y, x) )</td>
</tr>
</tbody>
</table>
Graph of the 5th Chebyshev Polynomial of the first type

\[ T_5(x) = 16x^5 - 20x^3 + 5x \] acting on \( \mathbb{Z}_{31} \).
Visualization $r = 2$

$\eta$ maps $(x, y, z) \rightarrow (z, y, xy - z)$ in $\mathbb{Z}_5$
Length of Maximum Orbits

The following plots depict the length of the maximum orbit versus the prime $p$ for a length two and length three word, respectively.
## Period Data

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
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<tbody>
<tr>
<td>p</td>
<td>L(p)</td>
<td></td>
<td>L(p)</td>
</tr>
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<td>------</td>
<td>---</td>
<td>------</td>
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<tr>
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<td>168</td>
</tr>
<tr>
<td>31</td>
<td>372</td>
<td>31</td>
<td>186</td>
</tr>
</tbody>
</table>
Questions

▶ Question: How many $\alpha$’s in $Out(F_2)$ are needed to make $\bigcup_\alpha \{\alpha^\kappa(x_{max})|\kappa \geq 0\} = \kappa^{-1}(\kappa(x_{max}))$?

▶ Find a pair of matrices that realize the dip. Explore why the form of those matrices gave us a dip in the first place.
Future Work

- Study maximum orbit lengths in order to learn a tight bound on the largest period while varying primes.
- Aim to formulate a similar study varying degree of $\mathbb{F}_q$ over $\mathbb{F}_p$ for a fixed prime $p$. 