The When and How:
Classifying the Generators of the Semigroups Associated with Berenstein-Zelenvinsky Graph Quiltings

Austin Alderete, Conor Nelson, Mark Tuben, Shams Alyusof

M.E.G.L. - Polytopes

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Objectives

- To introduce the semigroup $P_{\Gamma,N}$.
- To determine when it is possible to combinatorially describe elements of a generating set of $P_{\Gamma,N}$.
- To produce a description of the above when possible.

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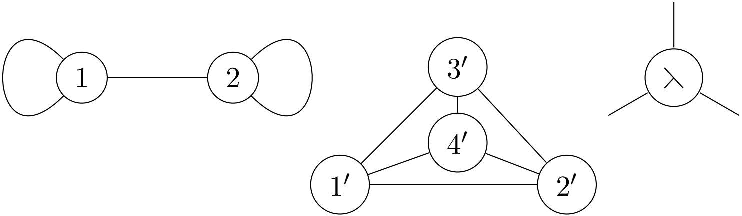
Trivalent graphs

Let $G = (V(G), E(G), \phi)$ such that for all $v \in V(G)$, $d_G(v) = 3$.
That is, we want G to be 3-regular general graph.

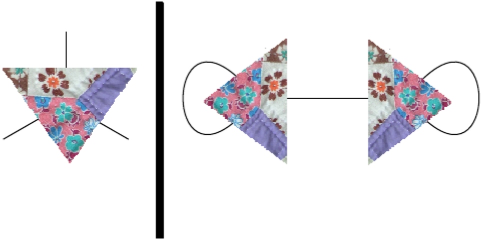
e.g.



We will also permit the graph we will call ⅄, which has one trivalent vertex and leaves.

Local structure and quilting

No matter which graph we choose, zooming in on any vertex we find that its local structure is that of $\leftthreetimes$. We are going to make a quilt out of our graph in the following way. At every vertex, place a triangle over the vertex like so:



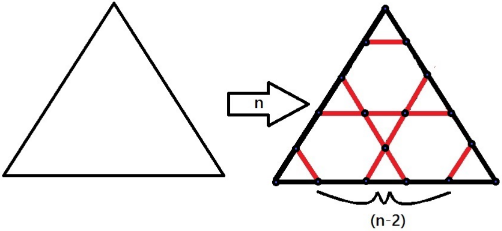
So, what exactly are these triangles?

Bernstein–Zelevinsky Triangles (BZ)

- Technically speaking, these are elements of the affine semigroup $BZ(SL_m(\mathbb{C}))$.
- They are precisely the integer points in the intersection of a polyhedral cone with an affine subspace.
- As far as we are concerned though (for the purposes of this presentation), each BZ triangle is simply an n-tuple that satisfies a set of rules laid out by the visual structure of the triangle and the set of all such n-tuples satisfying the rules forms a semigroup.

The structure of BZ triangles

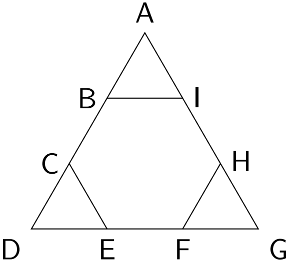
We first need to fix an $n \geq 2 \in \mathbb{N}$. To the triangle, we then add three sets of $(n-2)$ lines across the triangle in order to form the BZ triangle.



At each place where two lines intersect, we add a vertex. These vertices will be given weight and each weighted triangle is a BZ triangle.

The rules of BZ triangles

- Hexagon equalities. Within each honeycomb, we restrict equalities of the sums of opposite sides of the honeycomb. $[B+C=H+F, I+H=E+C, B+I=E+F]$
- The weightings of the triangles are elements in our semigroup.



Vertex Weightings

A

B I

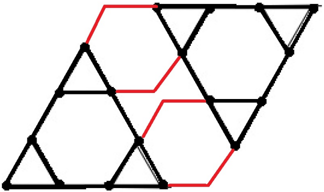
C H

D E F G

Element of Semigroup

Quilting Rules

What happens when we "quilt" two or more triangles together?
We generate a new hexagon in between the two triangles. Again, we restrict these hexagons to have the same equalities as described earlier.
NB! The hexagon rules for this new hexagon need only be satisfied on the black sides.



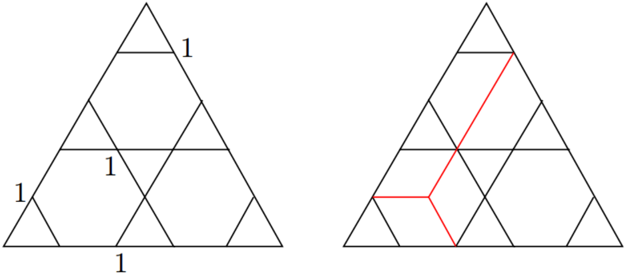
Representing elements of $BZ(SL_n(\mathbb{C}))$

Although we can simply represent each element as the $n-$tuple of vertex weightings or as the triangle with labeled vertex weights, it is often more pleasing, and advantageous, to represent elements by paths.

A path can simply be thought of as a graph overlaid on the BZ triangle whose edges track the weightings. This isn't exactly clear without an example...

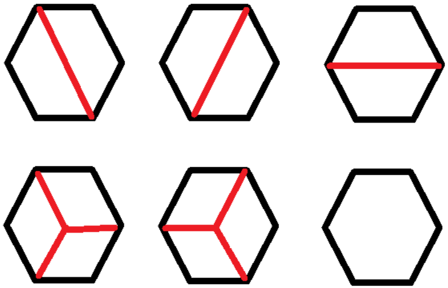
Paths

On the left is the BZ triangle while on the right is the path that represents it.



Possible paths within a hexagon

Within each hexagon, every valid path can be locally deconstructed into a direct sum of the following six local paths:



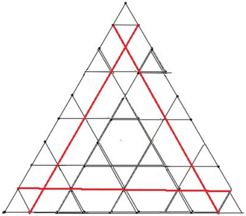
Decomposition of Paths

Recall that one of the objectives is to determine the generators of this semigroup. For an element to be a generator of the semigroup, it must be indecomposable. That is, it cannot be decomposed into a direct sum of representations. This is also a sufficient condition.

The obvious question is what does a decomposable path look like.

Example of decomposition

Observe as a path is decomposed by choosing a hexagon, removing a “strand” from it, and then removing more strands from other hexagons until the rules are once more satisfied.



Of course, it suffices to find only one hexagon which “unravels” the path to make it decomposable. More interestingly, if there is one hexagon whose “removal” removes the entire path, the path is indecomposable.

The semigroup

Definition: $P_{\Gamma,N}$

Let Γ be a 3-regular general graph and $N \geq 2$ be a natural number. Then $P_{\Gamma,N}$ is the semigroup whose elements are the weighted quilts composed of BZ triangles satisfying both the BZ and quilting rules.

NB! $P_{\Gamma,N}$ has the structure of the lattice points within a polyhedral cone.

Core Ideas:

- We wish to find the generators of our semigroups.
- The generators are precisely the indecomposable paths we can draw satisfying all rules.

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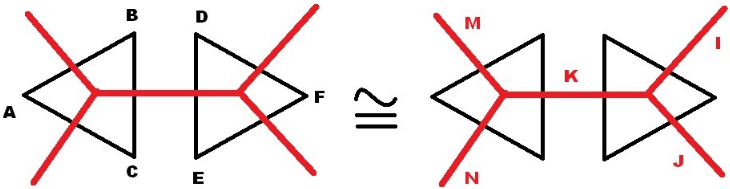
Special properties of $P_{\Gamma,2}$

We consider a special case of the BZ triangles. These are special triangles because they do not insist on hexagon rules. So the only inequalities that are forced are the triangle inequalities. Thus the quilting is of this form:

$$A + B = M, \quad A + C = N,$$
$$B + C = D + E = K$$
$$E + F = J, \quad D + F = I$$



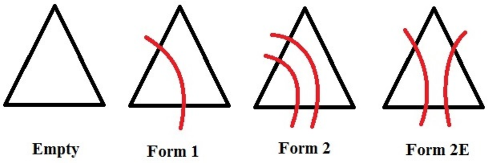
Special properties of $P_{\Gamma,2}$

The following was proven by Dr. Manon:
The subset of $P_{\Gamma,2}$ whose elements have weight less than or equal to 2 is a generating set.

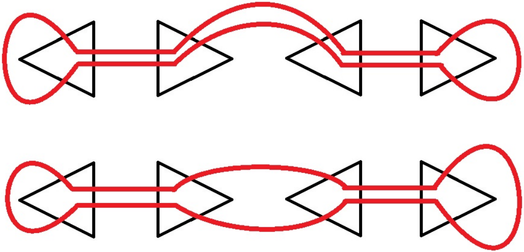
Theorem 1.1

Theorem 1.1

The generators of the $P_{\Gamma,2}$ semigroup corresponding to extremal rays in the associated polyhedral cone are precisely the paths which locally have only the following forms: form empty, 1, 2, 2E. If Form 2E occurs, it must occur exactly twice.



The following two paths are both found within the generating set. However, only one of these (the top one) is also an extremal ray in the associated polyhedral cone.



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Chaos motivation and statement

Unlike with the $P_{\Gamma,2}$ case, there is no proof that states that we can construct a generating set by selecting elements whose weights are less than a specific n. Therefore, we set out to prove that we could make indecomposable paths with arbitrary weightings.

Chaos Theorem

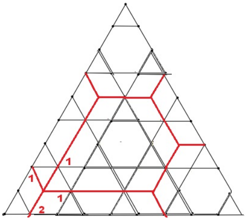
For every natural number m,

1) there exists some trivalent graph Γ such that $P_{\Gamma,3}$ has an indecomposable with maximum weight m
2) there exists some natural number N such that $P_{\curlywedge,N}$ has an indecomposable with maximum weight m.

We will show that (2) implies (1).

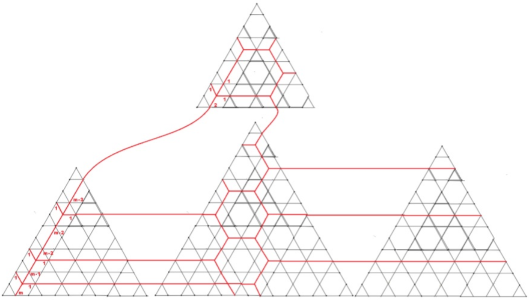
Proof of Chaos (2)

The proof of this is relatively straightforward and is done by giving an explicit construction of an indecomposable path with maximum weight m for any arbitrary but fixed natural number m. First, we need note that the following is indecomposable.

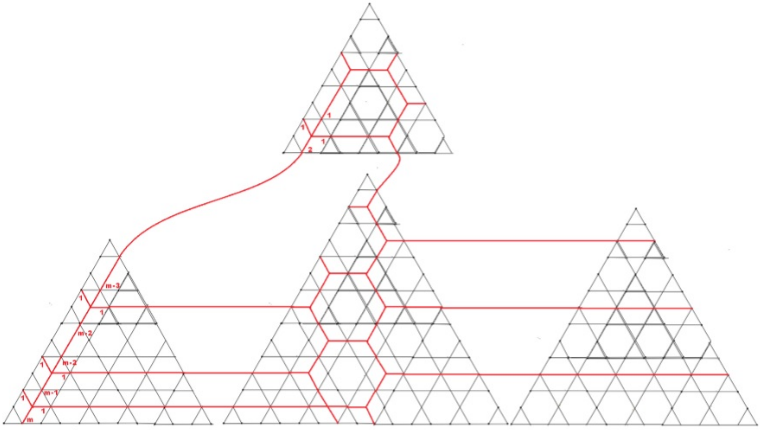


Proof of Chaos (2) (cont.)

The "keystone" from the previous slide stands at the top of this new, massive triangle. Its basic pattern is repeated again and again, each time adding a weight of $+1$ to the bottom left while remaining indecomposable. This is sufficient to prove (2).



Demonstration of the indecomposability of the construction



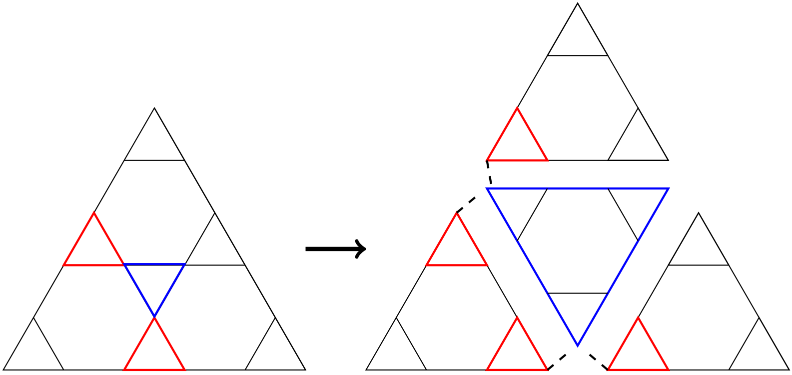
Creation of an algorithm

To prove (1), we constructed an algorithm that took any BZ triangle/path with maximum weight m and embedded the weightings in a quilt constructed from $n = 3$ BZ triangles while preserving indecomposability.

The algorithm should:

- Be an algorithm!
- Preserve decomposibility
- Preserve weightings

Algorithm (Basic idea)



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Summary and next steps

By proving Chaos, we have shown that, while the $P_{\Gamma,2}$ case is fairly tame, the $P_{\Gamma,N}$ case rapidly blows up, requiring that we consider much larger path weightings in order to produce a generating set. Future work may focus on:

- For a fixed N, can we put a bound on the weight needed?
- Is our generating set ever minimal, and if not, how do we find a minimal generating set?
- What are the extremal rays of the associated polyhedral cone for $P_{\Gamma,N}$?

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