Polytopes of Trivalent Spin Diagrams

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Overview:

Build: Create polyhedral cones from trivalent spin diagrams.

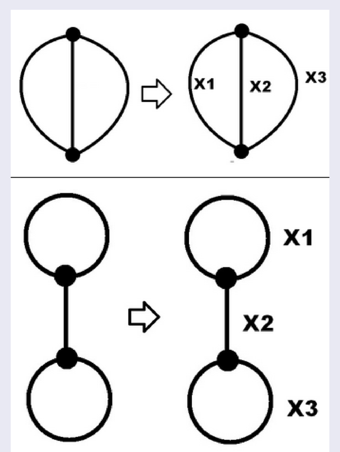
Identify: Find the extremal rays of generated polyhedral cones from the underlying diagrams

Classify: From the extremal rays of our generated cone and from a set of bounding half-spaces, construct the face poset of our polytope contained in the cone.

Trivalent Spin Diagrams: (Γ_w)

Definition: An abstract trivalent psuedograph whose edges e_i are assigned a weighting $x_i \in \mathbb{R}_{\geq 0}$.

- Γ_w can be represented as $(x_1, ..., x_d) \in \mathbb{R}^d$
- The x_i are subject to constraints (see below)



Polyhedral Generation:

The set $\mathfrak{P}_\Gamma$:

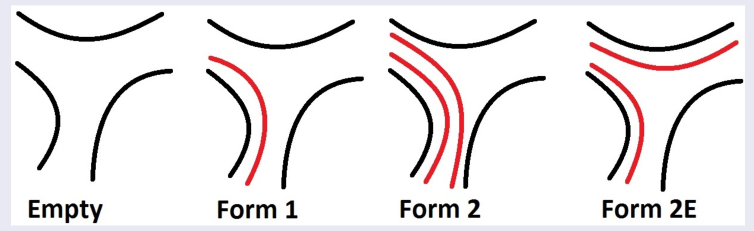
- The set of all Γ_w such that for each Γ_w, each x_i, x_j, x_k, at a node of Γ_w must satisfy the triangle inequalities.
- $\mathfrak{P}_\Gamma$ is an finite intersection of half-spaces (each of the form $A \cdot X \leq 0$) where each Γ_w is a point in this intersection.
- $\mathfrak{P}_\Gamma$, is a polyhedral cone.

The set $\mathfrak{P}_\Gamma(L)$:

- To create a polytope, we must bound our cone.
- We introduce leveling inequalities. At every node of Γ_w, $x_i + x_j + x_k \leq 2L$.
- $\mathfrak{P}_\Gamma(L)$ is the set of Γ_w satisfying the leveling and triangle inequalities.
- $\mathfrak{P}_\Gamma(L)$ is the polytope contained in the cone $\mathfrak{P}_\Gamma$.

Theorems

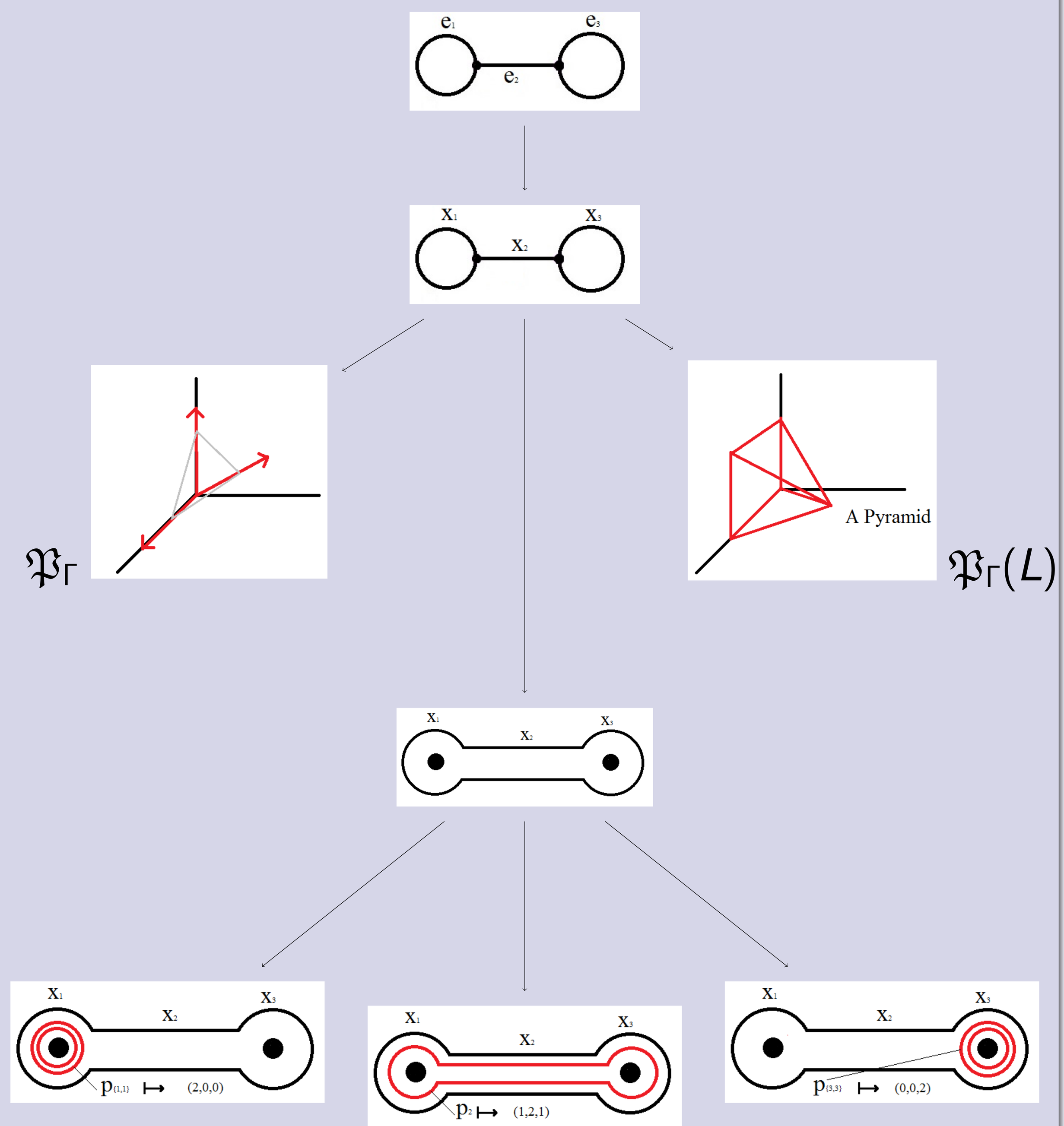
Theorem 1: Given some graph Γ, for any Spin diagram Γ_w, a path p on Γ_R generates an extremal ray R_E of the polyhedral cone $\mathfrak{P}_\Gamma$ whenever p is a simple planar path such that every node of Γ_R is of the form: Empty, Form 1, Form 2, or Form 2E, where Form 2E occurs never or exactly twice. Furthermore, every R_E can be generated from some simple planar path p where every node of Γ_R is of the form: Empty, Form 1, Form 2, or Form 2E, where Form 2E occurs never or exactly twice.



Theorem 2: Let $K = \{p_{E1}, .., p_{Ek}\}$ where p_{Ei} is a path or composition that generates a vertex v_i of $\mathfrak{P}_\Gamma(L)$ that lies on a R_E of $\mathfrak{P}_\Gamma$, and for any number $m \in \{2, 3, ..., k\}$ of p_{Ei}, their intersection is empty. Then $\bigcup_{i=1}^{m} p_i$ generates a new vertex v of $\mathfrak{P}_\Gamma(L)$ not generated from any individual $p_{Ei} \in K$.

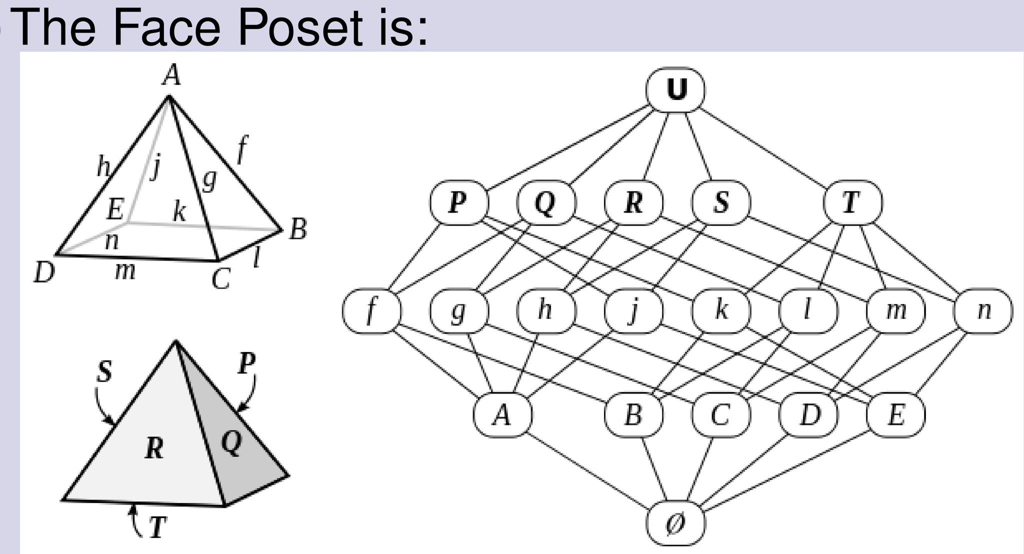
A Concise Breakdown of Our Γ called "Dumbbell"

Diagram Analysis



Further Analysis

- The respective sets of all nonnegative scalings of the points generated by $p_{\{1,1\}}$, p_2, & $p_{\{3,3\}}$ generates all three R_E of $\mathfrak{P}_\Gamma$.
- Γ_R with empty path $p_\emptyset$ generates the origin.
- Each path/composition described in left figure generates a vertex of $\mathfrak{P}_\Gamma(L)$.
- By Theorem 2, the point $(2, 0, 2)$ given by $p_{\{1,1\}} \cup p_{\{3,3\}}$ is a vertex of $\mathfrak{P}_\Gamma(L)$.
- The Face Poset is:

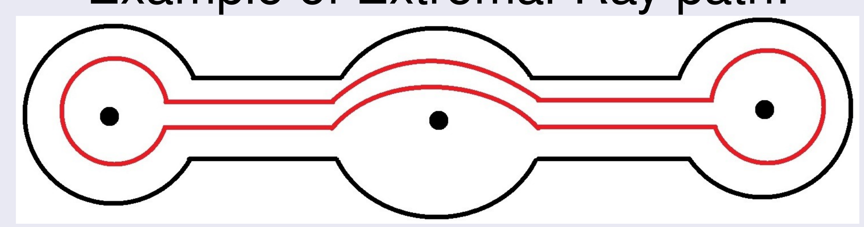


Paths, Extremal Rays, & Vertices

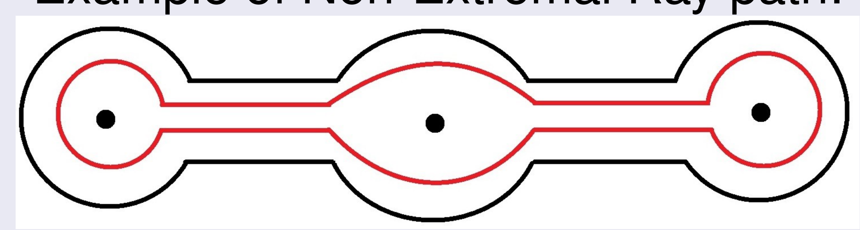
Paths/Compositions: Closed walks within thickened feynman diagrams (known as Ribbon Diagrams) Γ_R where the weightings x_i of Γ_w equal the number of times strands of a path p or composition of paths $p_{\{\}}$ crosses through e_i.

- With extremal rays R_E being unbounded edges of $\mathfrak{P}_\Gamma$, if a point Y lies on it, than R_E can be described as $\{\lambda Y | \lambda \geq 0\}$.
- So paths within Γ_R can generate point-representations of $R_E \in \mathfrak{P}_\Gamma$.

Example of Extremal-Ray path:



Example of Non-Extremal Ray path:



Future Work

- Further Investigate extremal ray behavior for different $\mathfrak{P}_\Gamma$.
- Learn more about the k-faces, where $k \in \{0, 1, 2, ..., d-1\}$, of the face posets of various $\mathfrak{P}_\Gamma(L)$.
- Understand the underlying semigroup algebra
- Study polytopes for other algebraic groups (SO_{2n}, SL_n, etc...)

Acknowledgments

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