Periods on Arithmetic Moduli Spaces

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Statement of the Problem
In this project, we study the dynamics of the action of several monoids/groups of morphisms of $\mathbb{F}_r$ (e.g. injections, general automorphisms, outer automorphisms) on the character variety $\text{Hom}(F_r, SL(2, \mathbb{F}_q))/\text{SL}(2, \mathbb{F}_q)[1]$. In particular, we characterize the orbits, provide criteria for determining periodic and preperiodic points, and compute the periods. We also work on visualizing the dynamics (orbits, functional graphs, etc.). We are concerned with $r \geq 1$ and $\mathbb{F}_q$ is of odd order.

We have classified the when the points of $\text{Hom}(F_r, SL(2, \mathbb{F}_q))/\text{SL}(2, \mathbb{F}_q)$ are periodic and preperiodic, and we have also begun to classify when the periods of the points of $\text{Hom}(F_r, SL(2, \mathbb{F}_q))/\text{SL}(2, \mathbb{F}_q)$.

Important Definitions

Definition ($\mathbb{F}_q$)
A finite field is a finite set on which the four operations -- multiplication, addition, subtraction and division (excluding by zero) are defined and satisfy the rules of arithmetic known as the field axioms. We denote a finite field of order $q$ by $\mathbb{F}_q$, and $\mathbb{F}_q$ its algebraic closure.

Definition (SL$_n$)
The special linear group of degree $n$ over a field $\mathbb{F}_q$ is the set of all $n \times n$ matrices with determinant 1 together with the operation of matrix multiplication. We denote this group by $\text{SL}_n(\mathbb{F}_q)$.

Definition (Dynamical System)
Let $S$ be a set and let $F: S \to S$ be a map from $S$ to itself. The iterate of $F$ with itself $n$ times is denoted $F^n = F \circ F \circ \cdots \circ F$.

A point $p \in S$ is periodic if $F^n(p) = p$ for some $n > 1$. The point is periodic if $\text{preperiodic}$ if $F^k(p)$ is periodic for some $k > 1$. The (forward) orbit of $p$ is the set $O_F(p) = \{p, F(p), F^2(p), F^3(p), \ldots \}$. Thus $P$ is preperiodic if and only if its orbit $O_F(p)$ is finite.

Definition (Conjugacy Equivalence)
We consider two matrices $A$ and $B$ to be equivalent if and only if, there exists a matrix $g$ such that $A = gBg^{-1}$. This forms an equivalence class of matrices.

The Future of the Problem
The next logical step in the problem, after we have classified the periods of $\text{SL}(2, \mathbb{F}_q)^3 / \text{SL}(2, \mathbb{F}_q)$ is to move onto the free group of 3 letters. Where we would examine the periods of $\text{SL}(2, \mathbb{F}_q)^3 / \text{SL}(2, \mathbb{F}_q)$. We know that the parametrization would look like: $[3]$

A = B, C → $T_1(x, y, z) = \lambda(x, y, z)$

Where $\text{SL}(2, \mathbb{F}_q)$ is to define a hyper-surface in $\mathbb{F}_q^3$ and any point on this hyper-surface will be defined by the traces of the three matrices.

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References


