Periods on Arithmetic Moduli Spaces

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Statement of the Problem

In this project, we study the dynamics of the action of several monoids/groups of morphisms of F_r (e.g. injections, general automorphisms, outer automorphisms) on the character variety $Hom(F_r, SL(2,\mathbb{F}_q))//SL(2,\mathbb{F}_q)$ [1]. In particular, we characterize the orbits, provide criterion for determining periodic and preperiodic points, and compute the periods. We also work on visualizing the dynamics (orbits, functional graphs, etc.). We are concerned with $r \geq 1$ and $\mathbb{F}_q$ is of odd order.

We have classified the when the points of $Hom(F_1, SL(2,\mathbb{F}_q))//SL(2,\mathbb{F}_q)$ are periodic and preperiodic, and we have also begun to classify when the periods of the points of $Hom(F_2, SL(2,\mathbb{F}_q))//SL(2,\mathbb{F}_q)$.

Important Definitions

Definition ($\mathbb{F}_q$)

A **finite field** is a finite set on which the four operations multiplication, addition, subtraction and division (excluding by zero) are defined, satisfying the rules of arithmetic known as the field axioms. We denote a finite field of order q by $\mathbb{F}_q$, and $\overline{\mathbb{F}}_p$ its algebraic closure.

Definition (SL_n)

The **special linear group of degree n** over a field $\mathbb{F}_q$ is the set of $n \times n$ matrices with determinant 1 together with the operation of matrix multiplication. We denote this group by $SL_n(\mathbb{F}_q)$.

Definition (Dynamical System)

Let S be a set and let $F : S \to S$ be a map from S to itself. The iterate of F with itself n times is denoted

$$F^{(n)} = F \circ F \circ \cdots \circ F$$

. A point $P \in S$ is **periodic** if $F(n)(P) = P$ for some $n > 1$. The point is **preperiodic** if $F(k)(P)$ is periodic for some $k \geq 1$. The (forward) orbit of P is the set

$$O_F(P) = \left\{P, F(P), F^{(2)}(P), F^{(3)}(P), \cdots\right\}.$$

Thus P is preperiodic if and only if its orbit $O_F(P)$ is finite.

Definition (Conjugation Equivalence)

We consider two matrices A and B to be equivalent if and only if, there exists a matrix g such that $A = gBg^{-1}$. This forms an equivalence class of matrices.

$r = 1$ The First Case

The trace of $A \in SL(2,\mathbb{F}_q)$ where A is diagonalizable, possibly over a field extension, parametrize $SL(2,\mathbb{F}_q) /\!\!/ SL(2,\mathbb{F}_q)$ as $\mathbb{F}_q$. Then for any $n \geq 1$ consider the power map $P_n : SL(2,\mathbb{F}_q) \to SL(2,\mathbb{F}_q)$ defined by $P_n([A]) = [A^n]$.

Let $t = \lambda + \lambda^{-1}$ and λ and λ^{-1} are the eigenvalues of A. The following are equivalent to P_n.

- $p_n : M_q \to M_q$ is given by $p_n([\lambda]) = [\lambda^n]$
- $T_n : \mathbb{F}_q \to \mathbb{F}_q$ is the nth Chebyshev polynomial of the first type defined recursively by $T_0(t) = 2$, $T_1(t) = t$, and $T_{n+2}(t) = tT_{n+1}(t) - T_n$.[2]

$r = 2$ The Second Case

The traces of the generators A, B, AB parametrize $SL(2,\mathbb{F}_q)^{\times 2} /\!\!/ SL(2,\mathbb{F}_q)$ as the affine space $\mathbb{F}_q^3$. The character map $Tr : SL(2,\mathbb{F}_q)^{\times 2} /\!\!/ SL(2,\mathbb{F}_q) \to \mathbb{F}_q^3$ given by

$$[\![A, B]\!] \mapsto (trA, trB, tr(AB))$$

is an isomorphism.

Actions on these traces are the Dynamical System we look at in this case. In this case all points are periodic.

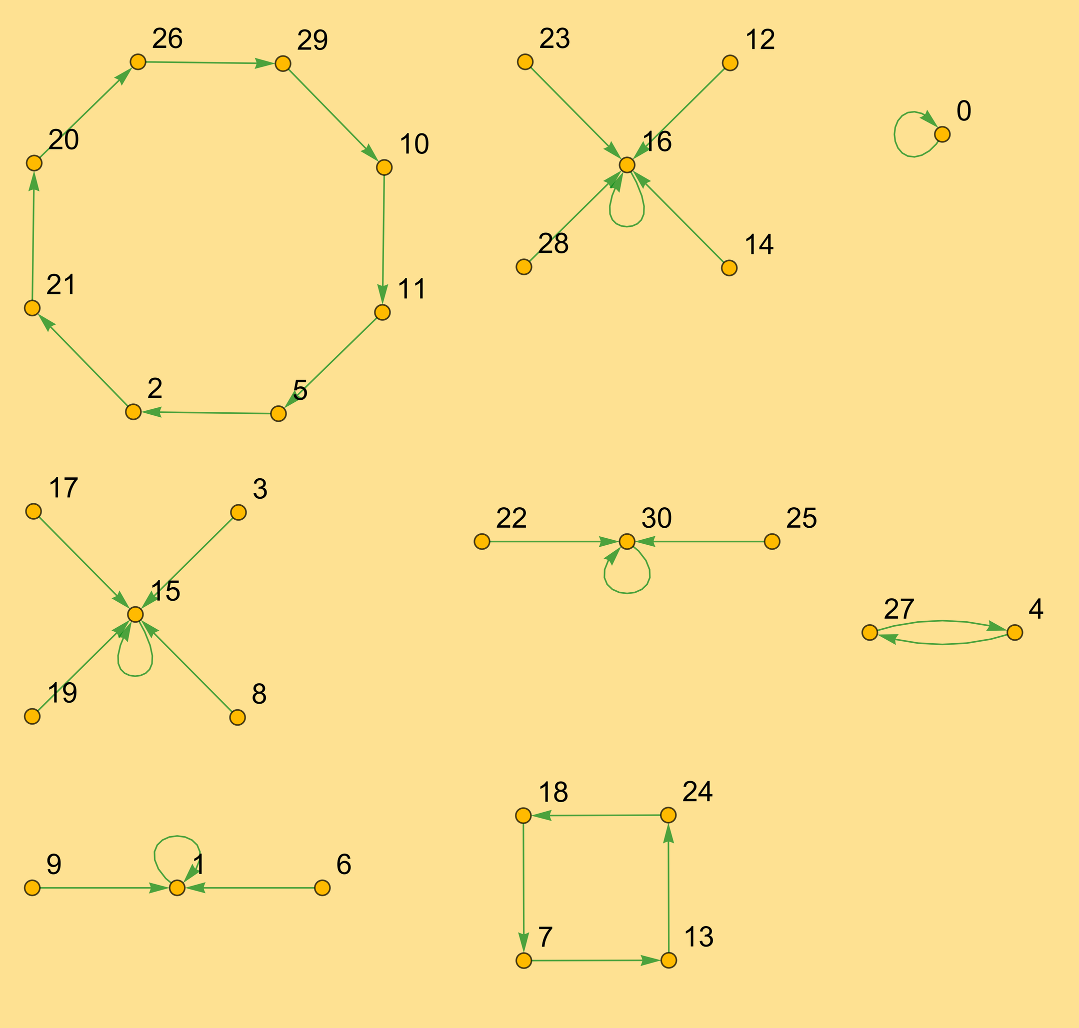
The action of $Out(F_2)$ on $SL(2,\mathbb{F}_q)$ induces an equivariant action on $\mathbb{F}_q^3$. Take $\nu = \tau\iota\eta^{-1}\iota\tau\iota$. The group generated by ι, η, τ is the same as generated by ι, η, ν. We get the following table.

	(A, B)	$(trA, trB, trAB)$	(x, y, z)
ι	(A^{-1}, B)	$(trA^{-1}, trB, trA^{-1}B)$	$(x, y, xy - z)$
τ	(B, A)	$(trB, trA, trBA)$	(y, x, z)
η	(AB, B)	$(trAB, trB, trAB^2)$	$(z, y, yz - x)$
η^{-1}	(AB^{-1}, B)	$(trAB^{-1}, trB, trA)$	$(xy - z, y, x)$
ν	(A^{-1}, AB)	$(trA^{-1}, trAB, trB)$	(x, z, y)

Visualization

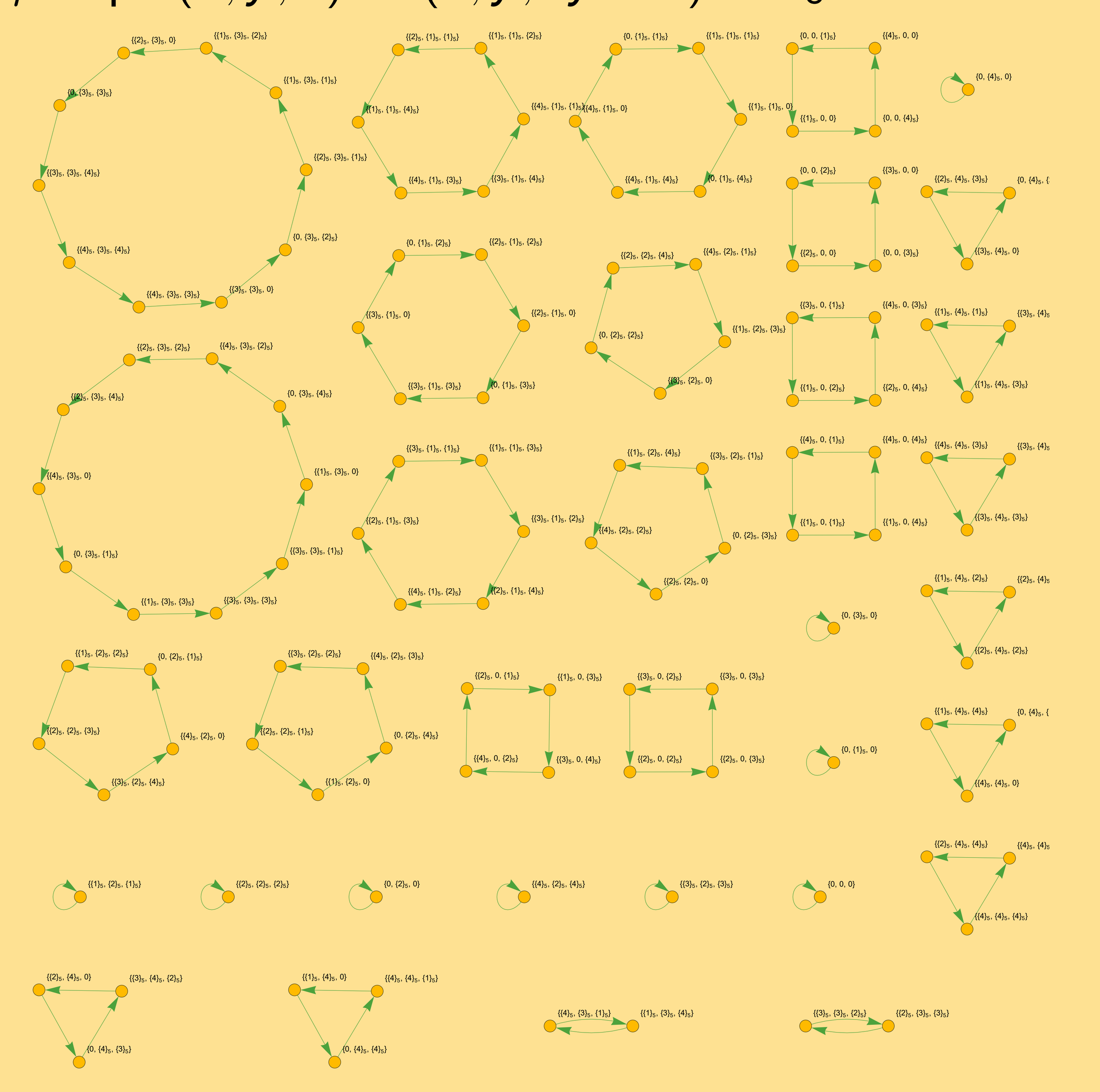
$r = 1$ Visualization

Graph of the 5th Chebyshev Polynomial of the first type $T_5(x) = 16x^5 - 20x^3 + 5x$ acting on $\mathbb{Z}_{31}$.



$r = 2$ Visualization

η maps $(x, y, z) \to (z, y, xy - z)$ in $\mathbb{Z}_5$



The Future of the Problem

The next logical step in the problem, after we have classified the periods of $SL(2,\mathbb{F}_q)^{\times 2} /\!\!/ SL(2,\mathbb{F}_q)$ is to move onto the free group of 3 letters. Where we would examine the periods of $SL(2,\mathbb{F}_q)^{\times 3} /\!\!/ SL(2,\mathbb{F}_q)$. We know that the parametrization would look like.[3]

$$(A, B, C) \mapsto$$

$$(Tr(A), Tr(B), Tr(C), Tr(BC), Tr(CA), Tr(AB), Tr(ABC)) \mapsto$$

$$(x_1, x_2, x_3, y_1, y_2, y_3, z) = (X, Y, z)$$

$$\Lambda(X, Y, z) = z^2 - p(X, Y)z + q(X, Y) = 0$$

Where $\Lambda(X, Y, z) = 0$ will define a hyper-surface in $\mathbb{F}_q^7$ and any point on this hyper-surface will be defined by the traces of the three matrices.

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References

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