Periods on Arithmetic Moduli Spaces

Diaaledin (Dia) Taha, Jermain McDermott, Robert Argus, Patrick Brown

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MEGL Symposium, August 2015

Outline

Periods on Arithmetic Moduli Spaces

- Setting the Stage
- The r = 1 Case
- The r = 2 Case
- The Big Picture and Future Work

Terminology and Notation

Periods on Arithmetic Moduli Spaces

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Definition (1.1)

The **free group** F_S over a given set S consists of all expressions (a.k.a. words, or terms) that can be built from members of S, considering two expressions different unless their equality follows from the group axioms.

Definition (1.2)

A **finite field** is a finite set on which the four operations multiplication, addition, subtraction and division (excluding by zero) are defined, satisfying the rules of arithmetic known as the field axioms.

Terminology and Notation (cont'd)

Periods on Arithmetic Moduli Spaces

Diaaledin (Dia) Taha, Jermain McDermott, Robert Argus, Patrick Brown Let \mathbb{F}_q be a finite field of order q.

Definition (1.3)

The general linear group of degree n over a field \mathbb{F}_q is the set of $n \times n$ invertible matrices together with the operation of matrix multiplication. We denote this group by $GL_n(\mathbb{F}_q)$.

Definition (1.4)

The special linear group of degree n over a field \mathbb{F}_q is the set of $n \times n$ matrices with determinant 1 together with the operation of matrix multiplication. We denote this group by $SL_n(\mathbb{F}_q)$.

Dynamical System

Periods on Arithmetic Moduli Spaces

Diaaledin (Dia) Taha, Jermain McDermott, Robert Argus, Patrick Brown Let S be a set and let $F: S \to S$ be a map from S to itself. The iterate of F with itself n times is denoted

$$F^{(n)} = F \circ F \circ \cdots \circ F$$

A point $P \in S$ is **periodic** if F(n)(P) = P for some n > 1. The point is **preperiodic** if F(k)(P) is periodic for some $k \ge 1$. The (forward) orbit of P is the set

$$O_F(P) = \left\{ P, F(P), F^{(2)}(P), F^{(3)}(P), \cdots \right\}.$$

Thus P is preperiodic if and only if its orbit $O_F(P)$ is finite.

The Setup

Periods on Arithmetic Moduli Spaces

Diaaledin (Dia) Taha, Jermain McDermott, Robert Argus, Patrick Brown Define $Onj(F_r) := Inj(F_r)/Inn(F_r)$, where $Inj(F_r)$ and $Inn(F_r)$ are the monomorphisms and inner automorphisms from the free group of rank r to itself, respectively. Consider

 $Q := Hom(F_r, SL_n(\mathbb{F}_q))/SL_n(\mathbb{F}_q)$ and let $Onj(F_r)$ act on Q.

The Process

- **1** Fix $[\alpha] \in Onj(F_r)$ and $[f] \in Q$.
- **2** Choose $\alpha' \in [\alpha]$ and $f' \in [f]$.
- **3** Compute $\alpha(f') := f' \circ \alpha'$.
- 4 Find $[\alpha'(f')] \in Q$ and iterate.

This defines a dynamical system. As \mathbb{F}_q is a finite field, it is reasonable to ask whether there exist periodic orbits.



The Moduli Space

Periods on Arithmetic Moduli Spaces

Diaaledin (Dia) Taha, Jermain McDermott, Robert Argus, Patrick Brown It turns out that $Hom(F_r, SL_n(\mathbb{F}_q))$ is a variety. One might then ask:

- I Is $Q = Hom(F_r, SL_n(\mathbb{F}_q))/SL_n(\mathbb{F}_q)$ a variety? **No**!
- 2 Does there exist an approximation to Q that is a variety \mathfrak{X} , where $\mathfrak{X} \subset Q$? **Yes**!
- **3** How do we find such a variety \mathfrak{X} ?

The variety $\mathfrak{X}:=[Hom(F_r,SL_n)//SL_n](\mathbb{F}_q)$ is obtained from Q by throwing out the strictly upper triangularizable matrices to obtain the following $Hom(F_r,SL_n(\mathbb{F}_q))^*/SL_n(\overline{\mathbb{F}}_q)$, which is a moduli space.

definitions

Periods on Arithmetic Moduli Spaces

- Denote $Q_{1,q} = \operatorname{Hom}(F_1, \operatorname{SL}(2, \mathbb{F}_q))/\operatorname{SL}(2, \mathbb{F}_q)$ (this is an instance of the set Q Robert introduced, the instance where r = 1).
- Note that this set $\mathfrak{X}_{F_1}(\mathrm{SL}(2,\mathbb{F}_q))$ is not the same as the set $\mathfrak{X}_{F_1}(\mathrm{SL}(2,\mathbb{F}_q)) = \mathrm{Hom}(\mathrm{F}_1,\mathrm{SL}(2,\mathbb{F}_q))//\mathrm{SL}(2,\mathbb{F}_q) \subset Q_{1,q}$ which discards "nondiagonalizable" elements.
- Also, define by $\mathcal{O}(\phi, \alpha)$ the dynamical system $\alpha(\phi), \alpha^2(\phi), \alpha^3(\phi), \dots$ ($\phi \in Q_{1,q}, \alpha \in \textit{Onj}(F_1)$ where F_1 is the free group of rank 1).
- Since $(F_1, \cdot) \cong (\mathbb{Z}, +)$ $(F_1$ is cyclic generated by a, Z is cyclic generated by 1), we have $\operatorname{Inn}(F_1) \cong \operatorname{Inn}(\mathbb{Z}) = \{\operatorname{id}_{\mathbb{Z}}\}$, the trivial group.



identifications

Periods on Arithmetic Moduli Spaces

- Again, F_1 is cyclic, so a homomorphism $\rho: F_1 \to G$ is determined by where a (the generator of F_1) is sent in G.
- From now on, identify an injective homomorphism $a \rightarrow a^n : n \in \mathbb{Z} \{0\}$ with n
- Similarly, we identify $\phi \in \operatorname{Hom}(F_1, \operatorname{SL}(2, \mathbb{F}_q))$ with A where $\phi(a) = A \in \operatorname{SL}(2, \mathbb{F}_q)$.
- We will identify $\gamma \operatorname{Inn}(F_1) = \{ \gamma \circ \operatorname{id}_{\mathbb{Z}} \} = \{ \gamma \} \in \operatorname{Onj}(F_1)$ by γ .

how to study Dyn

Periods on Arithmetic Moduli Spaces

- This turns $\mathcal{O}(\phi, \alpha)$ into $\mathcal{O}(A, n)$ where $\phi(a) = A$, $\alpha(a) = a^n$.
- Thus we have $\alpha(\phi), \alpha^2(\phi), \alpha^3(\phi), \dots$ equivalent to $A, A^n, A^{n^2}, A^{n^3}, \dots$
- Notice that in the set $Q_{1,q}$, we aren't dealing with $\phi \in Hom(F_1, \operatorname{SL}(2, \mathbb{F}_q))$ per se, but rather the conjugacy class of ϕ under the action of $\operatorname{SL}(2, \mathbb{F}_q)$.
- This identification of the conjugacy class of ϕ with the element ϕ will make sense after the following.

$\mathcal{P}: \mathcal{Q}_{1,q} \times \operatorname{Onj}(\mathcal{F}_1) \to \mathbb{N}$

Periods on Arithmetic Moduli Spaces

- We now want to define the map $\mathcal{P}: Q_{1,q} \times \operatorname{Onj}(F_1) \to \mathbb{N}$ by sending (ϕ, α) (considered as (A,n)) to the smallest positive number m if $\mathcal{O}(\phi, \alpha)$ has so called "prime" period m, or 0 if $\mathcal{O}(\phi, \alpha)$ isn't periodic.
- "Periodic" simply means there is a kth place in the sequence $A, A^n, A^{n^2}, A^{n^3}, \dots$ where we have $A^{n^k} = A$. The "period" of the system is the integer k.

finding the period

Periods on Arithmetic Moduli Spaces

- The reason we are identifying the conjugacy class of A with the element A is because the period is conjugate invariant.
- This comes from noting that $(gAg^{-1})^n = I_{2\times 2}$ iff $A^n = I_{2\times 2}$ for any $g \in SL(2, \mathbb{F}_q)$.
- We have shown in this case that the element A has order relatively prime to n iff A has a prime period.

Periods on Arithmetic Moduli Spaces

- The question now arises of orders of elements in $SL(2, \mathbb{F}_q)$.
- Two important facts mostly answer this question: that order and characteristic polynomial are also conjugate invariant; i.e., a conjugacy class of matrices has its own period and characteristic polynomial.

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- If the characteristic polynomial has no roots in \mathbb{F}_q , then it is diagonalizable over \mathbb{F}_{q^2} .
- Otherwise, our matrix is "parabolic", and has a double root in \mathbb{F}_q (given the matrix isn't $\pm I_{2\times 2}$, these are in their own class)

orders in $\mathrm{SL}(2,\mathbb{F}_q)$

Periods on Arithmetic Moduli Spaces

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The following table gives orders of elements in $SL(2, \mathbb{F}_q)$.

$\approx 10^{-10}$ m ≈ 10			
[Conjugacy class type	Representative	Order
ĺ	±Ι	$\begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}$	If I, 1; if -I, 2
	Parabolic ($lpha\in\mathbb{F}_q$, alpha $=1$ or $lpha eq\omega^2$)	$A = \begin{bmatrix} \pm 1 & \alpha * \\ 0 & \pm 1 \end{bmatrix}$	If $tr(A)=2$, $char(\mathbb{F}_q)$; if $tr(A)=-2$, $2char($
ĺ	Diagonalizable over \mathbb{F}_q	$\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$	$\frac{q-1}{\gcd(\bar{a}**,q-1)}$
	Diagonalizable over \mathbb{F}_{q^2}	$\begin{bmatrix} c & 0 \\ 0 & c^{-1} \end{bmatrix}$	$rac{q^2-1}{\gcd(ilde{c}***,q^2-1)}$ (will divide $q+1$)

In this table:

st lpha = 1 or is not a square in \mathbb{F}_q .

** $ar{a}$ represents $\phi(a)$ where $\phi:(\mathbb{F}_q,\cdot) o(\mathbb{Z}_{q-1},+)$

*** \tilde{c} represents $\gamma(c)$ where $\gamma:(\mathbb{F}_{q^2},\cdot)\to(\mathbb{Z}_{q^2-1},+)$ with ϕ,γ being isomorphisms.

Periods on Arithmetic Moduli Spaces

- We now know the orders of elements in $SL(2, \mathbb{F}_q)$. This along with the n corresponding to alpha determines whether \mathcal{P} returns 0 or not.
- This answers the question of whether $\mathcal{O}(\phi, \alpha)$ has a prime period.
- To find this prime period is equivalent to solving $n^k \equiv 1$ mod ord(A) for k.
- This yields that factors of $\phi(ord(A))$ are the only candidates for k (ϕ is the euler phi function).

Periods on Arithmetic Moduli Spaces

Diaaledin (Dia) Taha, Jermain McDermott, Robert Argus, Patrick Brown ■ In any group G with $a \in G$ of finite order k = |a|, a is periodic under $\tau_n(a) = a^n$ if and only if gcd(k, n) = 1 (assume n > 1).

Periods on Arithmetic Moduli Spaces

- In any group G with $a \in G$ of finite order k = |a|, a is periodic under $\tau_n(a) = a^n$ if and only if gcd(k, n) = 1 (assume n > 1).
- (←) If $\gcd(k,n)=1$, then by Euler's theorem, $n^{\phi(k)}=1$ mod k. Thus, in the sequence $A, A^n, A^{n^2}, ...$, since $A^{n^j}=A^m$ with $n^j=m$ mod k for any $j\in\mathbb{N}$, we will find A again in at most the $\phi(k)$ 'th position.

Periods on Arithmetic Moduli Spaces

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- (→) If A is periodic, assume $gcd(k,n) \neq 1$. Then n is a zero divisor modulo k. Since an element of a ring is a zero divisor if and only if it is not a unit, there is no $s \in \mathbb{N}$ such that $n^s = 1 \mod k$ or else n^{s-1} would be an inverse of n, making n a unit modulo k. This is a contradiction, hence gcd(k,n) = 1.

what is the set $Q_{r,q}$?

Periods on Arithmetic Moduli Spaces

- We used the language of an element of $Q_{1,q}$ as being "diagonalizable" because we could identify an equivalence class with a diagonalizable matrix in $SL(2, \mathbb{F}_q)$.
- In fact, for general r, we can identify an element of $Q_{r,q}$ to be an element in $\mathrm{SL}(2,\mathbb{F}_q)^r/\mathrm{SL}(2,\mathbb{F}_q)=$ $\mathrm{SL}(2,\mathbb{F}_q)\times\mathrm{SL}(2,\mathbb{F}_q)\times...\times\mathrm{SL}(2,\mathbb{F}_q)/\mathrm{SL}(2,\mathbb{F}_q).$
- To see how to identify $\operatorname{Hom}(F_r,\operatorname{SL}(2,\mathbb{F}_q))$ with $\operatorname{SL}(2,\mathbb{F}_q)^r$, send an element $\rho \to (\rho(w_1),\rho(w_2),...,\rho(w_r))$ where $w_1,...,w_r$ are the generators of F_r . This map is a bijection.

Moving to a Free Group of 2 Letters

Periods on Arithmetic Moduli Spaces

- We move from a single matrix $A \in SL_2(F_q)$ to $\{(A, B) : (A, B) \in SL_2(F_q)^2\}$ and several things change.
- We go from having q + 4 Conjugation Classes to $2q^3 + q^2 + 4q + 1$
- We change from Onj(F1) to Out(F2)

Out(F2)

Periods on Arithmetic Moduli Spaces

- Out(F2) is generated by 3 Nielson Transformation
- Denote the Nielson transformations on $F_2 = \langle x_1, x_2 \rangle$ as follows:

$$\iota = \begin{bmatrix} x_1 & x_2 \\ x_1^{-1} & x_2 \end{bmatrix},$$

$$\tau = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_1 \end{bmatrix}, \text{ and}$$

$$\eta = \begin{bmatrix} x_1 & x_2 \\ x_1 x_2 & x_2 \end{bmatrix}.$$

Out(F2) Cont.

Periods on Arithmetic Moduli Spaces

Diaaledin (Dia) Taha, Jermain McDermott, Robert Argus, Patrick Brown • Out(F2) is isomorphic to $GL_2(\mathbb{Z})$

$$\bullet \iota \approx \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Products of the Nielson Transformations

Periods on Arithmetic Moduli Spaces

$$1 \quad \iota^m = \iota^m \pmod{2}$$

$$2 \tau^m = \tau^m \pmod{2}$$

$$\eta^m = \begin{bmatrix} x_1 & x_2 \\ x_1 x_2^m & x_2 \end{bmatrix}$$

5
$$\iota \eta^m = \begin{bmatrix} x_1 & x_2 \\ x_1^{-1} x_2^m & x_2 \end{bmatrix}$$
 and $\eta^m \iota = \begin{bmatrix} x_1 & x_2 \\ x_2^{-m} x_1^{-1} & x_2 \end{bmatrix}$

6
$$\tau \eta^m = \begin{bmatrix} x_1 & x_2 \\ x_2 x_1^m & x_1 \end{bmatrix}$$
 and $\eta^m \tau = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_1 x_2^m \end{bmatrix}$

Action of $Aut(F_2)$ on $Hom(F_2, SL_2(\mathbb{F}_Q))$

Periods on Arithmetic Moduli Spaces

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For any $(A, B) \in SL(2, \mathbb{F}_q)$ and $m \in \mathbb{Z}$, the Nielson transformations act as follows:

$$\iota^{m} \cdot (A, B) = (A^{(-1)^{m}}, B),$$

$$\tau^{m} \cdot (A, B) = \begin{cases} (B, A) & m \text{ is odd} \\ (A, B) & m \text{ is even} \end{cases}, \text{ and}$$

$$\eta^{m} \cdot (A, B) = (AB^{m}, B).$$

Periods for $Out(\mathbb{F}2)$ Length 1

Periods on Arithmetic Moduli Spaces

Let
$$(A, B) \in SL(2, \mathbb{F}_q)^{\times 2}$$
.

- **1 Case** ι : Since $\iota^2 = id$, we have that $\mathcal{O}_{\iota}((A, B))$ is periodic, with period at most 2.
- **2** Case τ : Same as for ι .
- 3 Cases η and η^{-1} (and η^m for $m \in \mathbb{Z}$): Since $\#SL_2(\mathbb{F}_q) = q^3 q < \infty$ is a finite group, raising any element in it to its order gives the identity. We thus get $\eta^{\pm ord(B)} \cdot (A,B) = (AB^{\pm ord(B)},B) = (A,B)$. So $\mathcal{O}_{\eta}((A,B))$ is periodic with period $ord(B) \mid \#SL_2(\mathbb{F}_q)$. In fact, for any $m \in \mathbb{Z}$ we have that $\mathcal{O}_{\eta^m}((A,B))$ is periodic of period

$$\frac{ord(B)}{\gcd(m, ord(B))}.$$

Periods for $Out(\mathbb{F}2)$ ι and η

Periods on Arithmetic Moduli Spaces

Diaaledin (Dia) Taha, Jermain McDermott, Robert Argus, Patrick Brown Case for $\iota\eta$ and $\eta\iota$ (and $\iota\eta^m$ and $\eta^m\iota$ for $m\in\mathbb{Z}$) The period of (A,B) is

$$\mathcal{O}_{\iota\eta^m}((A,B)): (A,B) \to (A^{-1}B^m,B) \to (B^{-m}AB^m,B) \to (B^{-m}A^{-1}B^{m+1},B) \to \dots$$

$$(B^{-km}AB^{km},B) \to (B^{-km}A^{-1}B^{(k+1)m},B) \to \dots$$

Note that

$$(\iota \eta^m)^{2k} \cdot (A, B) = (B^{-km} A B^{km}, B) = (\sigma_{B^{-m}}(A), B),$$

and so $\mathcal{O}_{\iota\eta^m}((A,B))$ is periodic of period at most

$$2 \times \frac{ord(B)}{\gcd(ord(B), mk)}.$$

We get the same behavior for $\eta^m \iota$.

Periods for $Out(\mathbb{F}2)$ τ and η

Periods on Arithmetic Moduli Spaces

Diaaledin (Dia) Taha, Jermain McDermott, Robert Argus, Patrick Brown Case for $\tau\eta$ and $\eta\tau$

We have that

$$\eta \tau = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_1 x_2 \end{bmatrix}.$$

It can be easily seen that for any $(A,B)\in SL(2,\mathbb{F}_q)$ and $n\in\mathbb{N}$ that

$$(\eta\tau)^n((A,B))=(s_n,s_{n+1}),$$

where

$$s_0 = A,$$

 $s_1 = B,$ and
 $s_{n+2} = s_n s_{n+1}$ for $n \ge 0.$

Note that $(s_n)_{n\in\mathbb{N}}$ is the "Fibonacci sequence" on the group $SL(2,\mathbb{F}_q)$.

Periods on $Hom(F_2, SL_2(\mathbb{F}_Q))$

Periods on Arithmetic Moduli Spaces

Diaaledin (Dia) Taha, Jermain McDermott, Robert Argus, Patrick Brown We have the following fact about finite dynamical systems.

Proposition

Let X be a finite set, and $f: X \to X$ be a function. It follows that:

- 1 any element $x \in X$ is preperiodic, and
- **2** *if* f *is invertible, then every* $x \in X$ *is periodic.*

From the invertibility of the action of ι , τ , and η we get the following:

Corollary

Let G be a finite group, then under the action of $Aut(F_2) \curvearrowright Hom(F_2, G)$ every orbit is periodic.

Moving from $Hom(F_2, SL_2(\mathbb{F}_Q))/SL_2(\mathbb{F}_q)$ to $Hom(F_2, SL_2(\mathbb{F}_Q))//SL_2(\mathbb{F}_q)$

Periods on Arithmetic Moduli Spaces

- $Hom(F_2, SL_2(\mathbb{F}_Q))/SL_2(\mathbb{F}_q)$ is computationally intensive.
- lacksquare We have a better option $\mathit{Hom}(F_2,\mathit{SL}_2(\mathbb{F}_Q))//\mathit{SL}_2(\mathbb{F}_q)$
- This allows us to transition from $\{(A,B):A,B\in SL_2(\mathbb{F}_q)\}$ to $\{(x,y,z):x,y,z\in \mathbb{F}_q\ \&\ x=Tr(A),y=Tr(B),z=Tr(AB)\}$

Trace Functions

Periods on Arithmetic Moduli Spaces

Diaaledin (Dia) Taha, Jermain McDermott, Robert Argus, Patrick Brown

The traces of the generators X, Y, XY parametrize $SL(2, \mathbb{F}_q)^{\times 2} /\!\!/ SL(2, \mathbb{F}_q)$ as the affine space \mathbb{F}_q^3 .

Trace Functions

Periods on Arithmetic Moduli Spaces

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The traces of the generators X,Y,XY parametrize $SL(2,\mathbb{F}_q)^{\times 2} /\!\!/ SL(2,\mathbb{F}_q)$ as the affine space \mathbb{F}_q^3 . The character map $Tr: SL(2,\mathbb{F}_q)^{\times 2} /\!\!/ SL(2,\mathbb{F}_q) \to \mathbb{F}_q^3$ given by

$$\llbracket A, B \rrbracket \mapsto (trA, trB, tr(AB))$$

is an isomorphism.

Automorphisms

Periods on Arithmetic Moduli Spaces

Diaaledin (Dia) Taha, Jermain McDermott, Robert Argus, Patrick Brown The action of $Out(F_2)$ on $SL(2,\mathbb{F}_q)^{\times 2} /\!\!/ SL(2,\mathbb{F}_q)$ is generated by the three involutions

Automorphisms

Periods on Arithmetic Moduli Spaces

Diaaledin
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The action of $Out(F_2)$ on $SL(2,\mathbb{F}_q)^{\times 2} /\!\!/ SL(2,\mathbb{F}_q)$ is generated by the three involutions

$$\iota^* : [A, B] \mapsto [A^{-1}, B]
\tau^* : [B, A] \mapsto [A, B]
\nu^* : [A, B] \mapsto [A^{-1}, AB]$$

Automorphisms

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Automorphisms

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This induces the following polynomial automorphisms on the affine space \mathbb{F}_q^3

$$h: (x, y, z) \mapsto (x, y, xy - z)$$

$$u: (x, y, z) \mapsto (y, x, z)$$

$$v: (x, y, z) \mapsto (x, z, y)$$

Automorphisms

Periods on Arithmetic Moduli Spaces

Diaaledin (Dia) Taha, Jermain McDermott, Robert Argus, Patrick Brown The action of $Out(F_2)$ on $SL(2, \mathbb{F}_q)^{\times 2} /\!\!/ SL(2, \mathbb{F}_q)$ is generated by the three involutions

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In what follows, denote the group they generate $\langle h, u, v \rangle \leq Aut(\mathbb{F}_a^3)$ by $\Gamma_{\mathbb{F}_a}$.

Periods on Arithmetic Moduli Spaces

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Robert Argus,
Patrick

Study the dynamics of $\Gamma_{\mathbb{F}_q} \curvearrowright \mathbb{F}_q^3$

Periods on Arithmetic Moduli Spaces

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Study the dynamics of $\Gamma_{\mathbb{F}_q} \curvearrowright \mathbb{F}_q^3$

■ What is the structure of $\Gamma_{\mathbb{F}_q}$?

Periods on Arithmetic Moduli Spaces

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Study the dynamics of $\Gamma_{\mathbb{F}_q} \curvearrowright \mathbb{F}_q^3$

- What is the structure of $\Gamma_{\mathbb{F}_q}$?
- Any interesting invariants?

Periods on Arithmetic Moduli Spaces

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Study the dynamics of $\Gamma_{\mathbb{F}_q} \curvearrowright \mathbb{F}_q^3$

- What is the structure of $\Gamma_{\mathbb{F}_q}$?
- Any interesting invariants?
- What do the orbits/periods look like?

Periods on Arithmetic Moduli Spaces

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Note that $PSL(2,\mathbb{Z})$ is a $(2,3,\infty)$ Coxeter group. It all fits together!



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- **1** That $\Gamma_{\mathbb{F}_q} \leq Aut(\kappa)$.
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- This implies two things:
 - **1** That $\Gamma_{\mathbb{F}_q} \leq Aut(\kappa)$.
 - **2** That $\kappa^{-1}(t)$ partitions \mathbb{F}_q^3 into $\Gamma_{\mathbb{F}_q}$ -invariant surfaces.

So studying the dynamics $\Gamma_{\mathbb{F}_q} \curvearrowright \mathbb{F}_q^3$ is equivalent to studying the dynamics of $\Gamma_{\mathbb{F}_q} \curvearrowright \kappa^{-1}(t)$.



Interlude: The Markoff-Hurwitz Equation

Periods on Arithmetic Moduli Spaces

Diaaledin (Dia) Taha, Jermain McDermott, Robert Argus, Patrick Brown The equation

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So in reality, we are studying a finitary Markoff-Hurwitz equation

$$x^2 + y^2 + z^2 = xyz + k.$$

Upper Bound on the size of $\Gamma_{\mathbb{F}_q}$ -Orbits

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this gives us a $\sim q^2$ upper bound for the $\Gamma_{\mathbb{F}_q}$ -orbits.

One Letter: h, u, or v

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h	XY = 2Z	q^2
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$$\#\mathcal{O}_*(x,y,z) := egin{cases} 1, & (x,y,z) \in \mathit{Fix}(*) \\ 2, & (x,y,z) \notin \mathit{Fix}(*) \end{cases}$$

Two Letters: u and v

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They generate S_3 .

Two Letters: *u* and *v*

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*	Fix(*)	#Fix(*)
uv	X = Y = Z	q

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$$\#\mathcal{O}_{uv}(x,y,z) := \begin{cases} 1, & (x,y,z) \in \mathit{Fix}(uv) \\ 3, & (x,y,z) \notin \mathit{Fix}(uv) \end{cases}$$

Two Letters: h and u

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They generate $\mathbb{Z}_2 \times \mathbb{Z}_2$.

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*	Fix(*)	# <i>Fix</i> (*)
hu	$X = Y, Z = 2^{-1}XY$	q

Two Letters: *h* and *u*

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*	Fix(*)	#Fix(*)
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Periods on Arithmetic Moduli Spaces

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$$\#\mathcal{O}_{hv}(x,y,z) := \begin{cases} 1, & (x,y,z) \in \mathit{Fix}(*), \text{ else} \\ p, & x = \pm 2 \\ \text{div. of } q-1, & x^2-4 \text{ is a} \neq 0 \text{ quad. res.} \\ \text{div. of } q+1, & x^2-4 \text{ is not a quad. res.} \end{cases}$$

Oh, the Places You'll Go!

Periods on Arithmetic Moduli Spaces

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- Flesh out our understanding of the arithmetic dynamics of $\Gamma_{\mathbb{F}_q} \curvearrowright \mathbb{F}_q^3$.
 - What is the size of $\Gamma_{\mathbb{F}_q}$?
 - What exceptional subgroups does it have?
 - What is the exact size of the orbits?
 - When is the action transitive on a surface $\kappa^{-1}(t)$?
 - How many connected components can $\kappa^{-1}(t)$ have?
 -
- Start the same program for the action of $Out(F_3) \curvearrowright SL(2, \mathbb{F}_q)^{\times} /\!\!/ SL(2, \mathbb{F}_q)$.

Acknowledgment

Periods on Arithmetic Moduli Spaces

Diaaledin (Dia) Taha, Jermain McDermott, Robert Argus, Patrick

- Prof. Sean Lawton
- GMU, MEGL
- GEAR, NSF

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