Finding Extremal Rays of Spin Diagram Polytopes

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M.E.G.L. - Polytopes

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Contents

Polytope Definitions

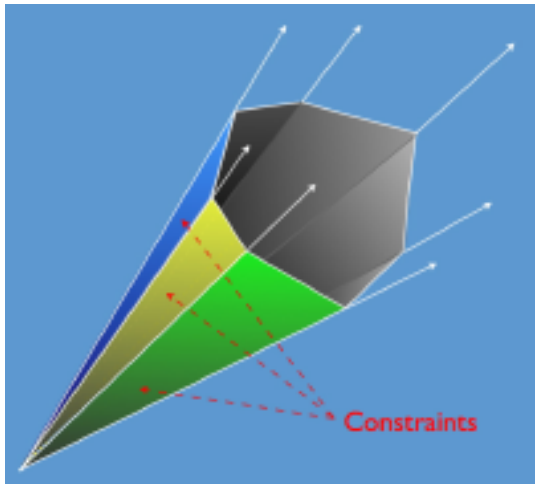
Polyhedral Set: A *polyhedral set* is an intersection of finitely many half spaces.

N.B.: *Polyhedral sets are convex.*

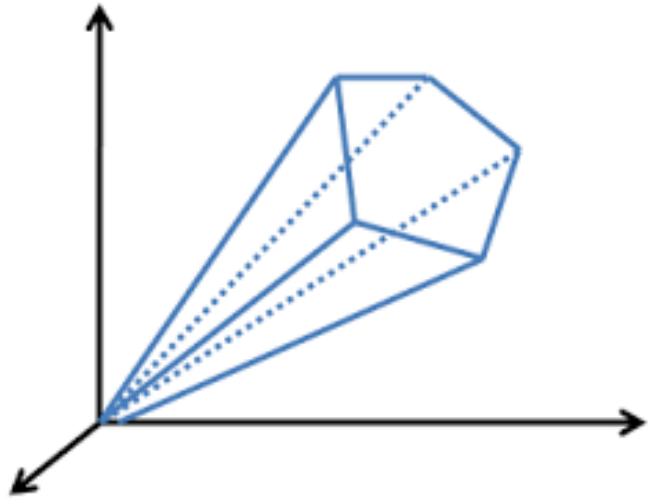
Polyhedral cone: An unbounded polyhedral set.

Polytope: A bounded polyhedral set.

Polyhedral cone versus polytope



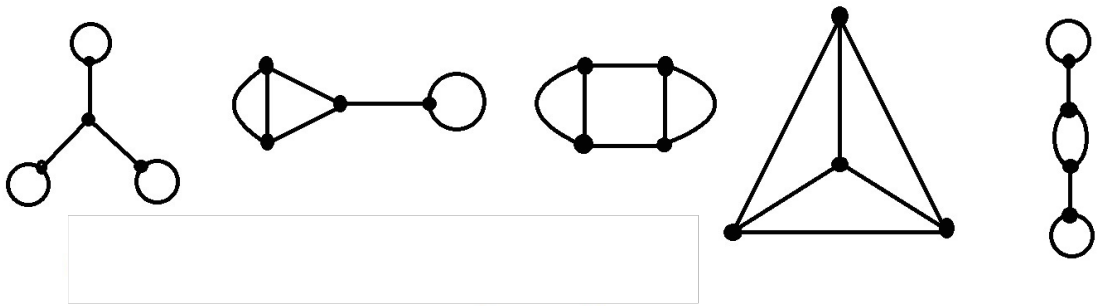
Unbounded polyhedral cone



Polytope

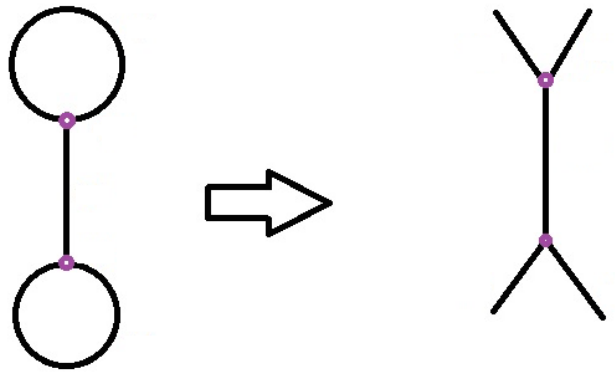
Feynman/Spin Diagrams

Graph Definition: We define Γ to be an abstract, finite, trivalent psuedograph. The set of nodes of Γ is denoted as $N(\Gamma)$, and the set of edges is denoted $E(\Gamma)$.



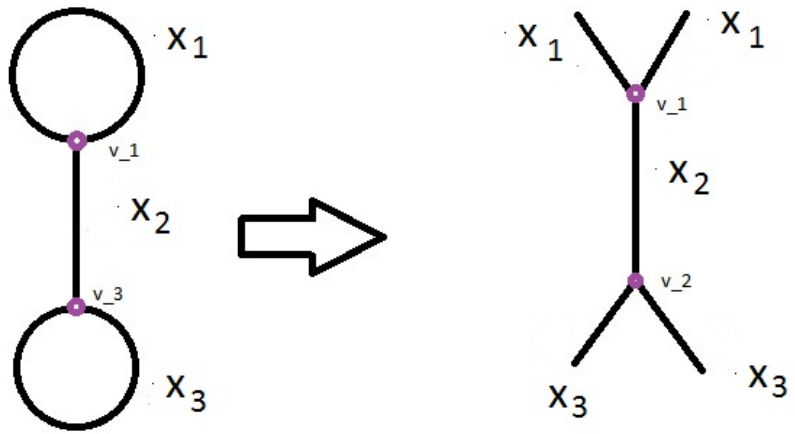
Feynman/Spin Diagrams

Note: *We consider a loop two edges as it connects to a node twice.*



Feynman/Spin Diagrams cont.

Feynman/Spin Diagram: A Feynman Diagram is supported on a graph, Γ, by assigning labels to edges that stand for non-negative, real numbers that satisfy inequalities.



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Triangle Inequality

Triangle Inequality

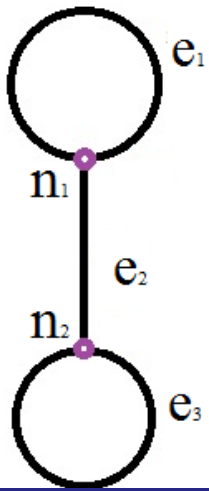
For any triangle, ABC, the sum of the lengths of any two sides of ABC must be greater than or equal to the length of the remaining side.

This holds true for degenerate triangles as well.

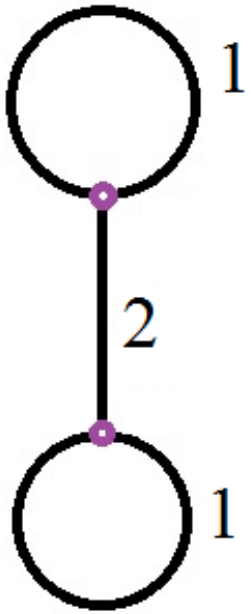
The set P_Γ

$$P_\Gamma = \left\{ (w(e_1), \ldots, w(e_d)) \mid e_i, e_j, e_k \text{ are connected to node } n_m \longleftrightarrow \begin{array}{l} w(e_i) \leq w(e_j) + w(e_k) \\ w(e_j) \leq w(e_k) + w(e_i) \\ w(e_k) \leq w(e_i) + w(e_j) \end{array} \right\}$$

- The set of all Feynman Diagrams (represented by ordered d-tuple of edge-weights), with underlying graph Γ.
- We consider the following Γ, which we will affectionately call 'Dumbbell':

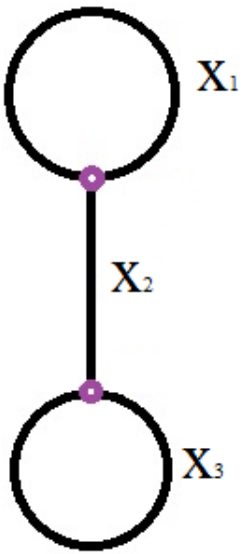


A single element of P_Γ for our choice of Γ is as follows



and is represented by $(1, 2, 1) \in P_\Gamma$. Note that all triangle inequalities are satisfied at every tri-node.

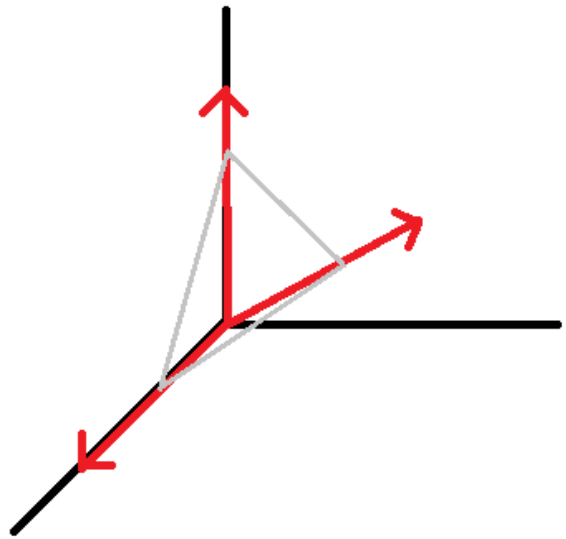
P_Γ is the set of all such varying weights $w(e_i) = x_i \in \mathbb{R}_{\geq 0}$



$(x_1, x_2, x_3) \approx$

given the edge-weights around tri-nodes satisfy the triangle inequalities.

P_Γ is an unbounded subset of $\mathbb{R}^3$ that satisfy a finite system of half-spaces (obtained from our triangle inequalities), which creates a unique polyhedral cone $\mathcal{P}$.

$P_\Gamma \to \mathcal{P} =$ 

Edges of the cone are the red arrows.

The set $P_\Gamma(L)$

We now introduce a new set $P_\Gamma(L) =$

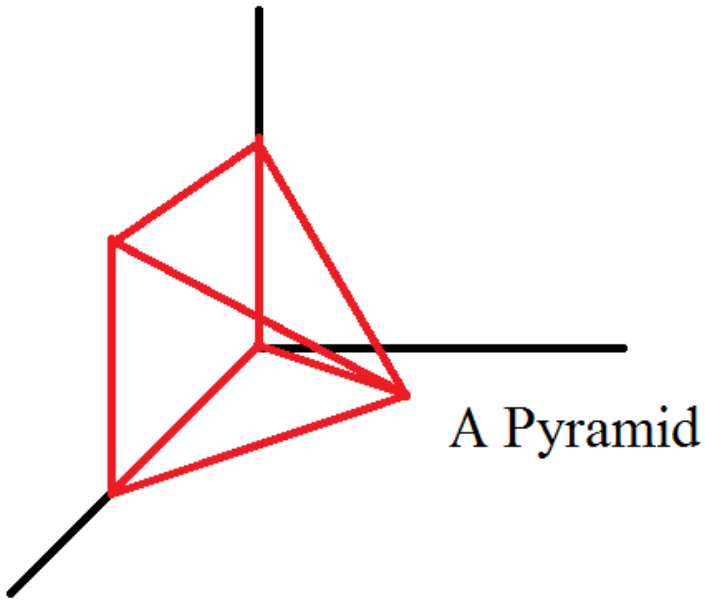
$\left\{w : E(\Gamma) \to \mathbb{R}_{\geq 0} \mid e_i, e_j, e_k \text{ are connected to node } n_m \longleftrightarrow w(e_i) + w(e_j) + w(e_k) \leq 2L\right\}$

where we require that $P_\Gamma(L) \subset P_\Gamma$.

It is clear that $P_\Gamma(L)$ is a bounded set, and in fact, is the polytope $P \subset \mathcal{P}$.

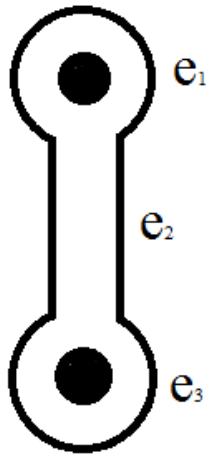
Considering the Γ Dumbbell again:

$P_\Gamma(L) \rightarrow \mathsf{P} =$



Ribbon Diagram

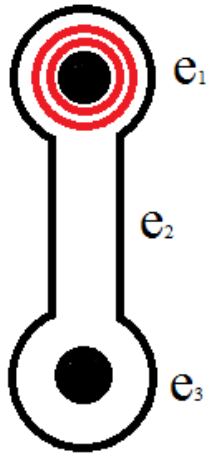
- We care about figuring out how we can describe points in $\mathcal{P}$ generated by the graphical analysis.
- Use these fattened version of our graphs Γ called ribbon graphs.
- Using the Γ Dumbbell, its fattened form is:



Paths & Compositions

- We can get certain points $X \in \mathcal{P}$ using **paths** p_i which are closed walks and each $w(e_i)$ is the number of times the path crosses through the fattened edge e_i.
- **Compositions** $p_{\{1,..,n\}}$ are a combination of paths $p_1, .., p_n$ where $w(e_i)$ is still equal to the total number times all the paths cross the edge e_i.

- Utilizing Dumbbell's fattened form:



The composition (in red) of two paths generate the point $(2, 0, 0) \in \mathcal{P}$

- We can also utilize paths to determine the edges of the polyhedral cone $\mathcal{P}$, which are also called the **Extremal Rays** (The red arrows in Dumbbell's $\mathcal{P}$ figure)

Contents

Simple Paths

What is a Simple Path?

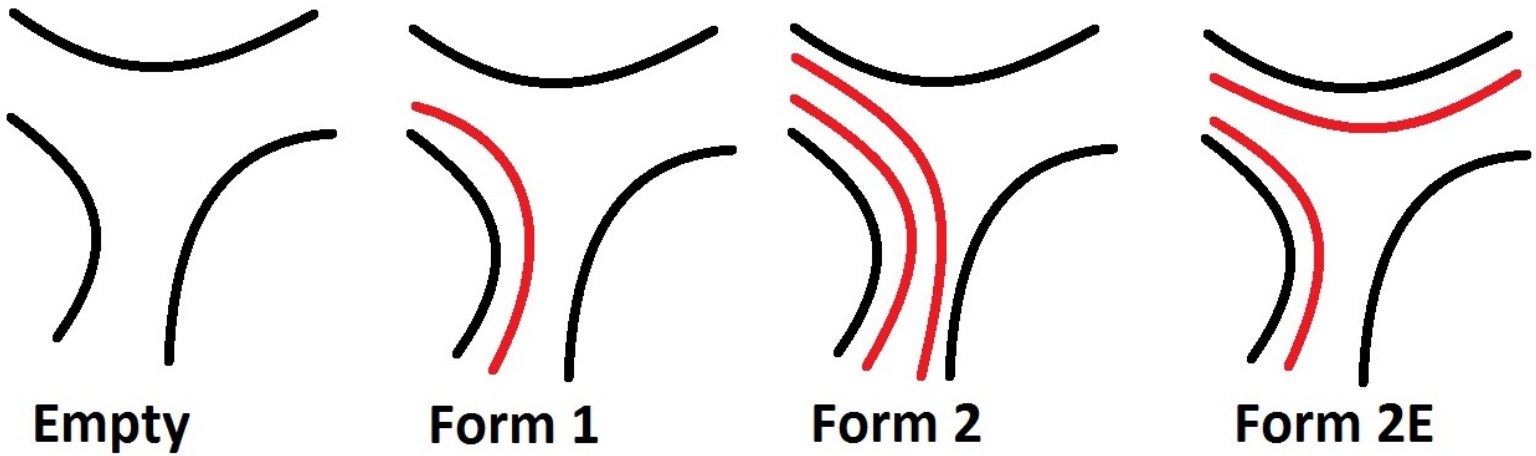
- Consider all paths of the thickened Feynman Diagrams (Ribbon Graphs).
- A simple path is a path on a Ribbon Graph with all edge weights being less than or equal to two. $\forall e_i$, $\text{str}(e_i) \leq 2$.
- The set of all simple paths is a subset of all paths of any given Feynman Diagram.
- Each simple path will generate a Ray. **More over, specific simple paths will generate Extremal Rays.**

Extremal Rays

Theorem

Theorem 1.1: For any Feynman diagram Γ, a path p on Γ_R generates an extremal ray of the polytope given by $\mathfrak{P}_R$ whenever p is a simple planar path such that every node of Γ_R is of the form: Empty, Form 1, Form 2, or Form 2E, where Form 2E occurs never or exactly twice. Furthermore, every extremal ray can be generated from some simple planar path p where every vertex of Γ_R is of the form: Empty, Form 1, Form 2, or Form 2E, where Form 2E occurs never or exactly twice.
A.A., J.C., C.M, C.N, M.S., C.T.

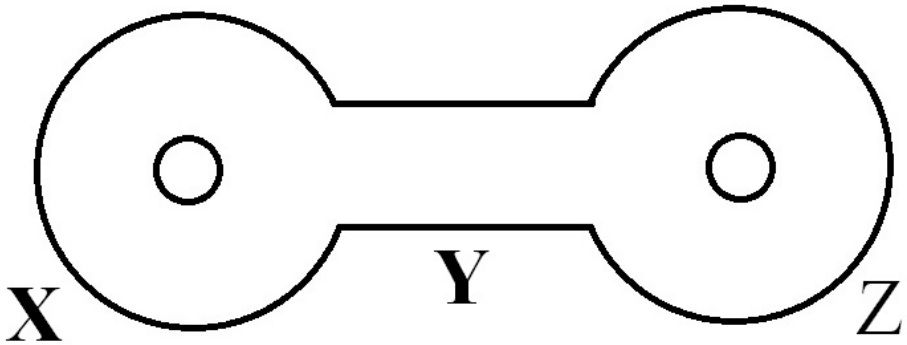
Restatement of Theorem 1.



All extremal rays can be generated from paths whose nodes are of these forms. With either exactly zero or two Form 2E's. Furthermore, whenever you have a path whose nodes are of these forms, you have an extremal ray.

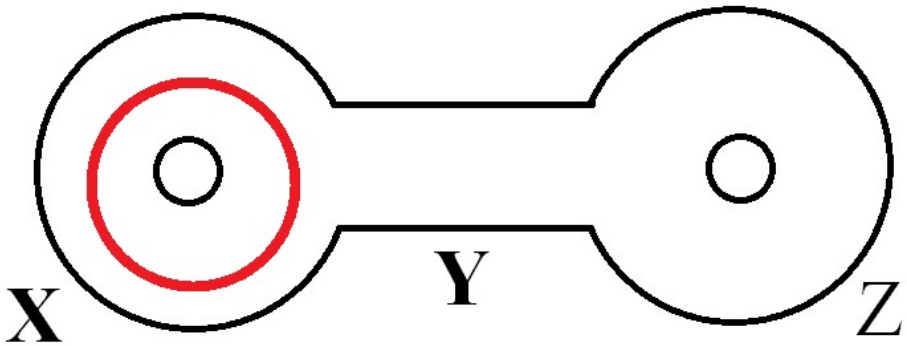
Example of Dumbbell

We are given the Ribbon Diagram of Dumbbell



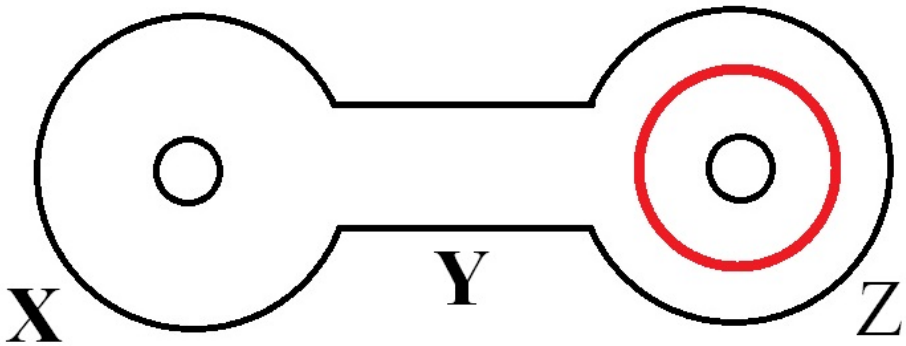
The empty path will generate the degenerate extremal ray.
(0,0,0)
Dumbbell's Extremal Rays: { (0,0,0) }

Draw a path around X.



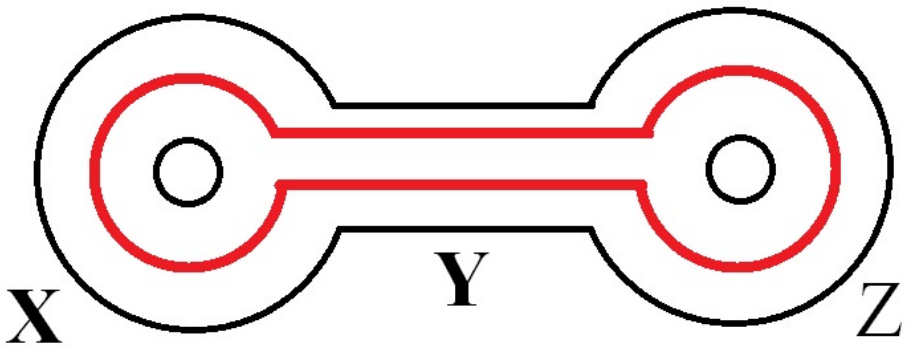
This path will generate the extremal ray with vertex representation: (2,0,0)
Dumbbell's Extremal Rays: { (0,0,0),(2,0,0) }

Draw a path around Z.



This path will generate the extremal ray with vertex representation: (0,0,2)
Dumbbell's Extremal Rays: { (0,0,0),(2,0,0),(0,0,2) }

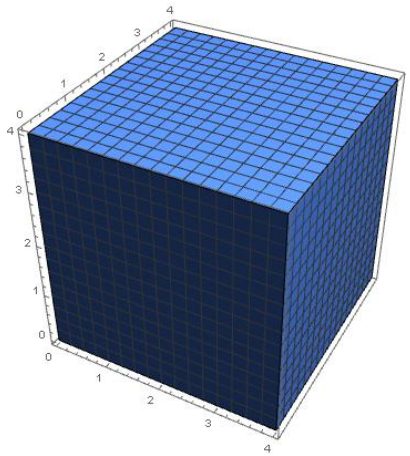
Draw a path that encompasses the whole graph.



This path will generate the extremal ray with vertex representation: (1,2,1)
Dumbbell's Extremal Rays: { (0,0,0),(2,0,0),(0,0,2),(1,2,1) }

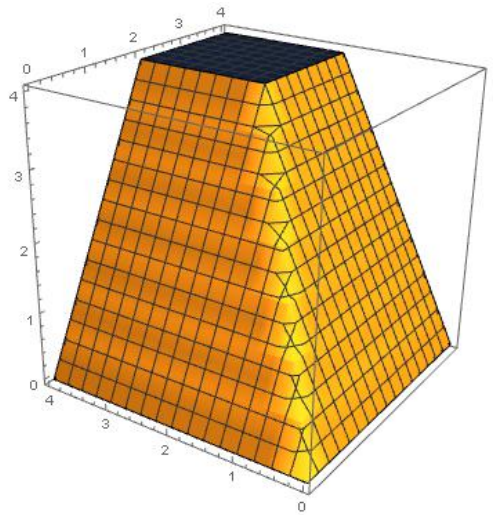
Intersection of Half-Spaces Example for Dumbbell

We will use Mathematica to give a visual representation of Dumbbell.
Consider the Positive Orthant:



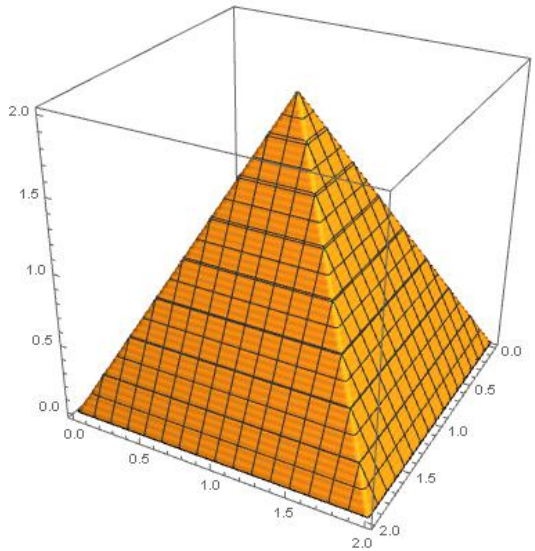
$$P_{Dumbbell} = \{X \geq 0, Y \geq 0, Z \geq 0\}$$

Now we will add the inequalities generated by $\mathfrak{P}_\Gamma$.



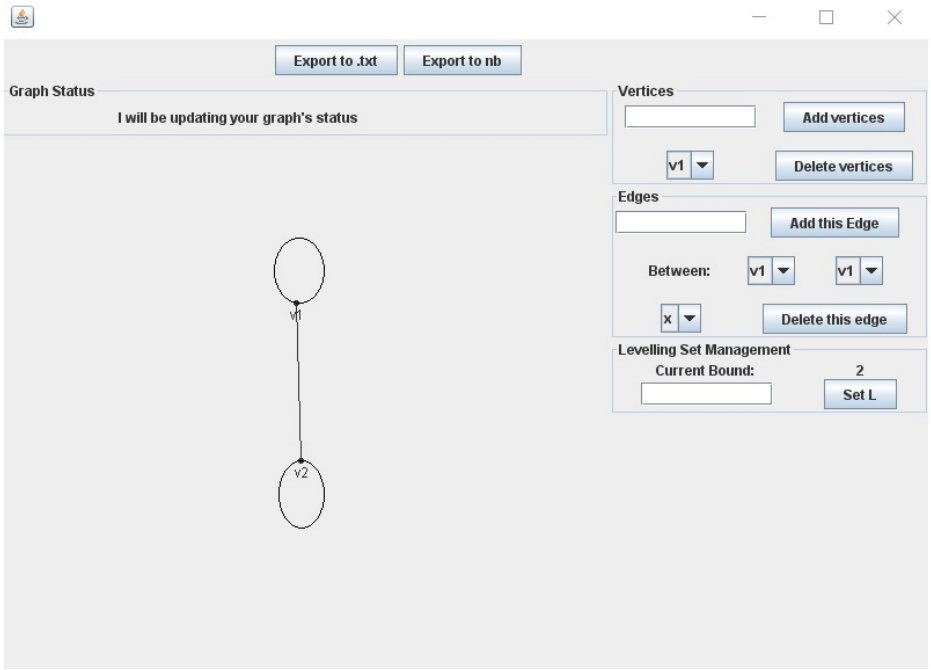
$$P_{Dumbbell} = \{X \geq 0, Y \geq 0, Z \geq 0, 2X \geq Y, 2Z \geq Y\}$$

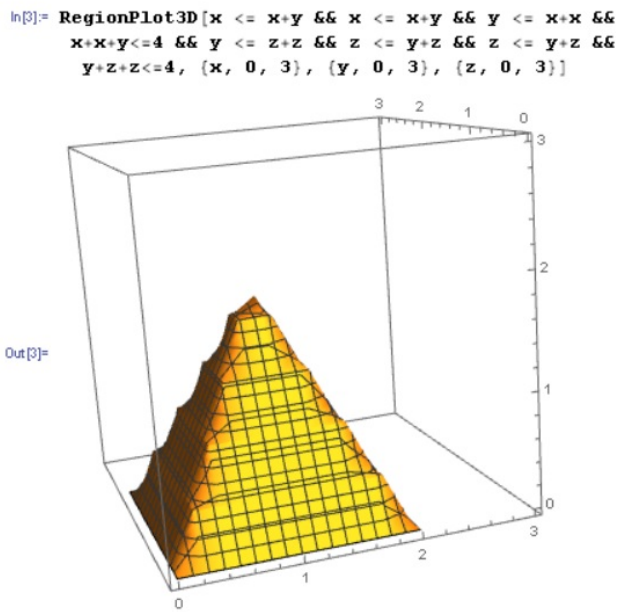
Finally, we will include the inequalities generated by $\mathfrak{P}_\Gamma(L)$.



$$P_{Dumbbell} = \{X \geq 0, Y \geq 0, Z \geq 0, 2X \geq Y,$$
$$2Z \geq Y, 2X + Y \leq 4, 2Z + Y \leq 4\}$$

We have also created a Graphical User Interface (GUI) to draw and interpret these diagrams. As shown below.





Semigroup Algebra Connection

- P_Γ can tell us about the algebraic geometry and the symplectic geometry of the character variety $X_g(\mathrm{SL}_2)$.
- $P_\Gamma(\mathrm{L})$ can be used to study more general moduli spaces and conformal field theory.
- For more info:
  1) Brew a pot of coffee
  2) See Dr. Manon

Contents

Find the vertices on extremal rays

How to find some of the vertices of P from the paths?

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1. We have a point on the ray R from the path p that generates R.

Find the vertices on extremal rays

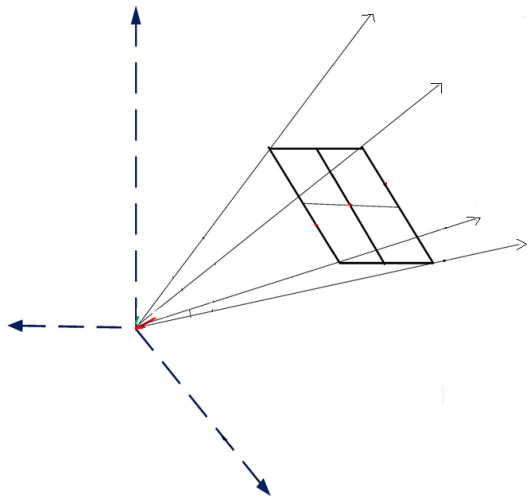
How to find some of the vertices of P from the paths?

1. We have a point on the ray R from the path p that generates R.

2. We need to find the vertex v on R with respect to w.

$$\text{i.e.,} \qquad \text{if} \quad \lambda(c_1, ..., c_d) = v \quad \text{then, } \lambda = ??$$

Leveling



3. Use the leveling equality to find the vertex.

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4. If p has a node with sum of weights 4, then $\lambda = L/2$ otherwise $\lambda = L$.

$$\lambda(c_i + c_j + c_k) = 2L$$

3. Use the leveling equality to find the vertex.

4. If p has a node with sum of weights 4, then $\lambda = L/2$ otherwise $\lambda = L$.

$$\lambda(c_i + c_j + c_k) = 2L$$

5. By varying the paths we can find all vertices on the extremal rays.

New vertices from old ones

Theorem

Let $K = \{p_1, p_2, ..., p_k\}$ be a set of paths that generate extremal rays of P such that $p_i \cap p_j = 0$ for all $i \neq j$. Then any subunion $p_{i_1} \cup ... \cup p_{i_t}$ where $p_{i_j} \in K$ represents a new vertex of $P_\Gamma(L)$.

Sketch of Proof:

1. Any subunion represents an exterior point v of the P.

2. Let $mv = v_1 + v_2 ... + v_m$ where v_i are vertices of P (using an algebraic property of the semigroup).

3. Levelling equality and mutually exclusive property of p_i forces each $v_j = v$.

Edges from extremal rays

1. The convex hull of $\{origin, v_i\}$ where v_i is a vertex formed from the path of an extremal rays.

2. The line segment joining v_i and $v_i + v_j$ such that $p_i \cap p_j = 0$ where p_i is the path that corresponds to the vertex v_i.

Contents

Future Work

Future work for MEGL - Polytopes Group:

- Attempt to create the k-faces of the Face Poset.
- Learn more about the face lattices of $\mathfrak{P}_\Gamma$ and $\mathfrak{P}_\Gamma(L)$.
- Understand the underlying algebra
- Study polytopes for other algebraic groups ($SO(2n), SL_n$, etc...)