Special Words in Free Groups

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Notation Used in the Presentation

- $SL_n\mathbb{C}$ Represents a $n \times n$ matrix whose elements are in the complex numbers and which has a determinant of 1.
- $GL_n\mathbb{C}$ Represents a $n \times n$ matrix whose elements are in the complex numbers and which has a determinant not equal to 0.
- $\mathbb{F}_r$ Represents a free group of r generators.

Free Groups

- A group is a set whose elements are closed under some operation, have an identity value, have an inverse value, and are associative.
- A free group is a group whose elements, known as words, are strings of characters that are the group's generators.
- The operation, which the Free Group is closed under, is concatenation.
- For example, the Free Group of rank 2 is the group of all strings with the characters a, b, a^{-1}, b^{-1}

Cyclic and Conjugate Equivalence

- A pair of words is cyclic if any of the two words can be rotated by taking the last element and moving it to the front to form the other word.
- A pair of words (w_1, w_2) is conjugate if they can be written as $w_1 = pw_2p^{-1}$ for some word p.
- Cyclic equivalence $\Longleftrightarrow$ Conjugate Equivalence
- This is because concatenation is the operation of the free group. wuw^{-1} is cyclically equivalent to $w^{-1}wu$, which is equal to u.

Word map

- Take $w \in \mathbb{F}_r$

$$w : (SL_n\mathbb{C})^r \to SL_n\mathbb{C}$$

$$(A_1, ..., A_r) \mapsto w(A_1, ..., A_r)$$

In other words, it replaces every occurence of the generator a_i of the free group by its corresponding matrix A_i. The product of all of those gives one matrix.

Trace function

- Take $w \in \mathbb{F}_r$

$$Tr_n(w) : (SL_n\mathbb{C})^r \to \mathbb{C}$$

$$(A_1, ..., A_r) \mapsto Tr(w(A_1, ..., A_r))$$

The question

We remark that conjugate words will always have the same trace function, the question is now, to which extent, the reverse of this implication is true.

Special Words

- Two words (or more precisely their conjugacy classes), $u, v \in \mathbb{F}_r$, are n-special if u is not conjugate to v and their trace functions (in $SL_n\mathbb{C}$) are equal. We say that they form a n-special pair.
- The set of special pairs for given n and r will be noted $\mathcal{S}_{n,r}$.
- A n-special word $u \in \mathbb{F}_r$ is a word which is in some n-special pair.

Abuse of notation

For sake of simplification we will talk about "words", however one should understand that this means "conjugacy classes of words".

Most of our further algorithms require that the words are written under the following form :

$$a^{\alpha_1} b^{\beta_1} \dots a^{\alpha_s} b^{\beta_s}$$

Actually one can verify that if a word is neither a^k nor b^l it is conjugate to a word of this form.

Special in $\mathbb{F}_2$ = Special in $\mathbb{F}_r$

- This fact allows us to concentrate on words of 2 generators.
- $S_{n,r} \neq \emptyset \iff S_{n,2} \neq \emptyset$

Higher Order Specials Are Contained In $\mathcal{S}_2$

- The sets of higher order special words, if they exist, will have the following characteristic:

$$\mathcal{S}_1 \supset \mathcal{S}_2 \supset \mathcal{S}_3 \supset \mathcal{S}_4 \supset \mathcal{S}_5 \supset \ldots \mathcal{S}_n$$

  Each set of special words is fully contained in the lower order sets.
- This is because any matrix $A \in SL_{n-1}$ matrix can be contained in an SL_n matrix.

Proof.

Suppose a matrix $A \in SL_{n-1}$. There exists a SL_n matrix constructed as:

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

□

The Intersection of Higher Special Orders is Empty

Another interesting fact is the following, all non-conjugate pairs won't have the same trace function in some $SL_n\mathbb{C}$, in other words :

$$\bigcap_{n=2}^{\infty} \mathcal{S}_n = \emptyset$$

Fricke Polynomial and Fricke Algorithm

- For every $w \in \mathbb{F}_2$, there exists a unique polynomial $P_w \in \mathbb{Z}[X, Y, Z]$, called the Fricke Polynomial. For all $A, B \in SL(2, \mathbb{C})$, $tr(w(A, B)) = P_w(tr(A), tr(B), tr(AB))$.
- The polynomial is a result of using the trace relation $(tr(AB) = tr(A)tr(B) - tr(AB^{-1}))$ to create a reduction algorithm that would take a word and break it down, and evaluate the Fricke polynomial of the word.

Fricke Polynomial Example

- Example: $tr(a^4bab) = tr(a^4b)tr(ab) - tr(a^3)$

$$tr(a^2)tr(a^2b) - tr(b)$$

$$tr(a)tr(ab) - tr(b^{-1})$$

Reverse Pairs are Special

A reverse word is given by taking the elements of a word from last to first.
$r(w)(a, b) = w(a^{-1}, b^{-1})^{-1}$

Example:
$w(a, b) = ababbba$
$r(w(a, b)) = (a^{-1}b^{-1}a^{-1}b^{-1}b^{-1}b^{-1}a^{-1})^{-1}$
$r(w(a, b)) = abbbaba$

Reverse Pairs are Special

Proof.

Suppose a word $W(a,b) \in \mathbb{F}_2$

$$\begin{aligned} Tr(W(a,b)) &= P_W(Tr(a),\ Tr(b),\ Tr(a,b)) \\ &= P_W(Tr(a^{-1}),\ Tr(b^{-1}),\ Tr(a^{-1}b^{-1})) \\ &= Tr(W(a^{-1},b^{-1})) \\ &= Tr(\overleftarrow{W(a,b)}) \end{aligned}$$

P_W represents the Fricke Polynomial of the trace. If they are not equal, then they must also be special with each other. $\square$

Inverse Pairs are Special

Using the characteristic polynomial as applied to $SL(2,\mathbb{C})$:

Proof.

$$\begin{aligned} X^2 - tr(X)X + I &= O \\ X^{-1}(X^2 - tr(X)X + I) &= X^{-1}O \\ tr(X) - tr(tr(X)I) + tr(X^{-1}) &= tr(O) \\ tr(X) - 2tr(X) + tr(X^{-1}) &= 0 \\ tr(X) &= tr(X^{-1}) \end{aligned}$$

□

SL Special Implies GL Special

Proof.

Any matrix, $A \in GL_n\mathbb{C}$, can be transformed into an SL matrix by multiplying the matrix by the scalar:

$$\sqrt{\frac{1}{Det(A)}}$$

Since special words must have the same exponent, the scalars changing the GL matrices to SL matrices of special words will be equal. Since trace is linear, the traces will remain equal after the scalar multiplications. Therefore, if two words are special in $SL_2\mathbb{C}$, they are special in $GL_2\mathbb{C}$ $\square$

GL Special Implies *SL* Special

Proof.

Since $SL_2\mathbb{C} \subset GL_2\mathbb{C}$, if a words are special in $GL_2\mathbb{C}$ then it is special in $SL_2\mathbb{C}$. $\square$

Therefore, words are special in $SL_2\mathbb{C}$ if and only if they words are special in $GL_2\mathbb{C}$.

(w, w^{-1}) is not 3-special

The proof involves two tools. First by standard analysis of the characteristic polynomial for $A \in SL_3\mathbb{C}$, we have :

$$\chi_A(X) = X^3 - tr(A)X^2 + tr(A^{-1})X - 1$$

Using the Cayley-Hamilton theorem we get that :

$$A^3 - tr(A)A^2 + tr(A^{-1})A = I_3$$

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$$A^3 - tr(A)A^2 + tr(A^{-1})A = I_3$$

Multiplying by A^{-1} and taking the trace we get :

$$2tr(A^{-1}) = tr(A)^2 - tr(A^2)$$

(w, w^{-1}) is not 3-special

On the other hand, using a lemma from Borel, we know that for any non-trivial word w the image of the word map is dense in $SL_3\mathbb{C}$. hence the closure of the set $\{w(A, B) \mid A, B \in SL_3\mathbb{C}\}$ is $SL_3\mathbb{C}$.

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If we take a non-trivial word w which is 3-special with its reverse then we will have :

$$2tr(w) = 2tr(w^{-1}) = tr(w)^2 - tr(w^2)$$

And using the density result, this should hold for any matrix $A \in SL_3\mathbb{C}$:

$$2tr(A) = tr(A)^2 - tr(A^2)$$

Since this is false, (w, w^{-1}) cannot be 3-special.

$(w, r(w))$ pairs are not 3-special ?

Since the inverse pairs won't be 3-special and reverse word construction is strongly related to the inverse :

$$r(w) = (w(a^{-1}), b^{-1}))^{-1}$$

We would expect that the reverse pairs are never 3-special pairs.

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The problem is more complicated because it may happen that $tr(w) = tr(r(w))$ namely when w and $r(w)$ are conjugate to each other.

The positive conjecture

It is also strongly conjectured that if we find an element $(w, w') \in \mathcal{S}_3$ then we can construct a new 3-special pair whose words are positive, i.e. with all powers to be positive.

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Hence we will search 3-special pairs within the set of 2-special pairs which are positive and non-reverse. [Law14] [LLM13]

Generating all Non-Cyclically Equivalent Words

- The function cycles through decreasing amounts of the largest exponent of a.
- The letters after that section of a's are permuted.
- The permutations are all joined to the section of a and these different concatenations are all the non cyclically equivalent words.
- Few cyclically equivalent words are produced in this process, so the time taken to delete them is minimal.

Generating Special Words

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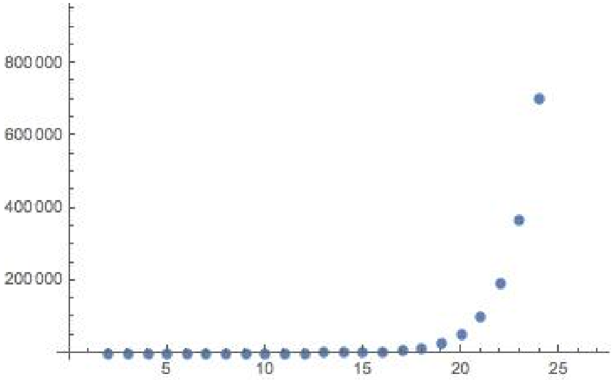
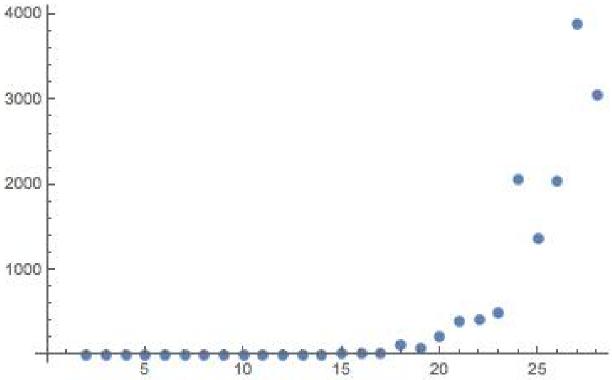
- The words with matching traces are collected together
- They words are checked if they are reverses and if so they are added to the list of specials
- If not, they are checked with another trace function and added to lists of all specials and all non reverse specials

Some statistics: number of conjugacy classes for a given length



Some statistics: number of words which are very special for a given length



Some statistics: probability to be very special for a given length

Table: table of ratios for length 14 to 21

Length	Number of words	Number of very special words	Ratio
14	1,182	4	0.003...
15	2,192	16	0.007...
16	4,116	20	0.004...
17	7,712	24	0.003..
18	14,602	111	0.007...
19	27,596	72	0.002...
20	52,488	224	0.004...
21	99,880	400	0.004...

Some statistics: probability to be very special for a given length

Table: table of ratios for length 22 to 28

Length	words	very special words	Ratio
22	361,505	416	0.00115...
23	1,008,650	496	0.000491...
24	1,963,021	2,072	0.00105...
25	3,755,152	1,368	0.000364...
26	12,343,581	2,054	0.0001664...
27	23,823,824	3,884	0.000163...
28	45,243,568	3,056	0.0000675...

Generating all words

The idea to generate all words of length n is the following, first generate all possible partitions of n into $2s$ positive numbers (those will represent the absolute values of the exponents for a word $a^{\alpha_1} b^{\beta_1} \dots a^{\alpha_s} b^{\beta_s}$) then multiply those partitions by all possible signs of those exponents and then delete conjugates. We are doing it for any s. This will give the set of all conjugacy classes for a given length (except a^n and b^n)

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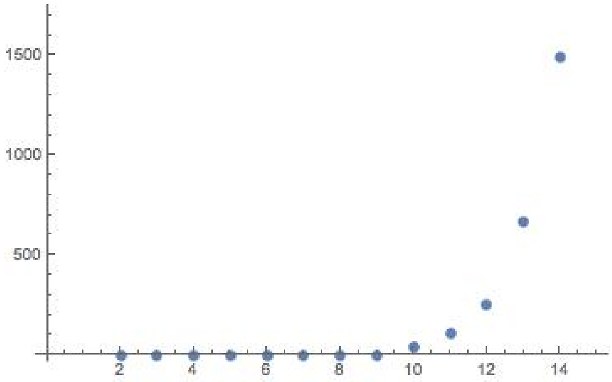
Once we have done this we apply the same sieves to get 2-special pairs. In our particular setting, to avoid calculus we delete reverse, inverse and inverse-reverse pairs from the data.

Some statistics: number of special classes of words



Representing Words as Matrices

- All n-Special pairs of words can be written in the form of $A^{\alpha_1}B^{\beta_1}.....A^{\alpha_k}B^{\beta_k}$.
- We use the exponents from each A and B as coordinates for a matrix.

Example of a Pattern Found With Matrices

- One pattern in the matrices we found was the matrix:

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ n & 0 & 0 \end{pmatrix}$$

- Where $n = 1$ at Length of 15 and increases by 1 for every 4th length.

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- Where $n = 1$ at Length of 15 and increases by 1 for every 4th length.
- This matrix will always produce 2-Special pairs:

$$ab(a^3b)^k a^2 b^2 ab(a^3b)^{n-k} a^2 b \tag{1}$$
$$ab(a^3b)^{n-k} a^2 b^2 ab(a^3b)^k a^2 b \tag{2}$$

- Where k lies between 0 and $\left\lfloor \frac{n+1}{2} \right\rfloor - 1$

Proof the Pair is Special

Breaking (1) and (2) up we get:

$$f = bab(a^3b)^k a^2 b$$

$$w_1 = bab(a^3b)^{n-k} a^2, \quad w_2 = ab(a^3b)^{n-k} a^2 b$$

Using the Trace relation:

$$Tr(AB) = Tr(A) * Tr(B) - Tr(AB^{-1})$$

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Using the Trace relation:

$$Tr(AB) = Tr(A) * Tr(B) - Tr(AB^{-1})$$

$$Tr(f)(Tr(w_1) - Tr(w_2)) - Tr(fw_1^{-1}) + Tr(fw_2^{-1})$$

Since, w_1 and w_2 are cyclic to one another, they will have equal trace functions. So we only have to see if $-\mathrm{Tr}(fw_1^{-1}) + \mathrm{Tr}(fw_2^{-1}) = 0$.

Proof the Pair is Special

By simplifying the last 2 terms in (2) we find:

$$\begin{aligned} fw_1^{-1} &= bab(a^3b)^k a^2ba^{-2}(a^3b)^{k-n}b^{-1}a^{-1}b^{-1} && (3)\\ &= a^2ba^{-2}(a^3b)^{2k-n} && (4)\\ &= a^2bab(a^3b)^{2k-n-1} && (5)\end{aligned}$$

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and,

$$\begin{aligned} fw_2^{-1} &= bab(a^3b)^k a^2 bb^{-1}a^{-2}(a^3b)^{k-n}b^{-1}a^{-1} && (6)\\ &= bab(a^3b)^{2k-n}b^{-1}a^{-1} && (7)\\ &= a^2bab(a^3b)^{2k-n-1} && (8)\end{aligned}$$

Thus, $fw_1 \sim fw_2$ and the pair is 2-Special.

Other Patterns in Matrix Format

$$\begin{pmatrix} n & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} n & n \\ n & 1 \end{pmatrix} \begin{pmatrix} n & 0 \\ 0 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ & . & & & \dots & 0 \\ & . & & & \dots & 0 \\ & . & & & \dots & 0 \\ 0 & n & 0 & & \dots & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ & . & & & . & . \\ & . & & & . & . \\ 1 & 1 & 0 & 0 & . & . \\ 1 & 0 & 1 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & & \dots & 0 \\ & . & & & . & 0 \\ & . & & & . & 0 \\ 0 & 2 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \end{pmatrix}$$

The Pairs Associated to the Preceding Matrices

- $(a^2bab^2)^n(ab)^na^2b^2$
  $(ab)^n(a^2bab^2)^na^2b^2$
- $a^nb^2ab^3abab^2ab$
  $a^nb^2abab^2ab^3ab$
- $(a^2b^2)^2aba^2b^2(ab)^{n-1}$
  $a^2b^2ab(a^2b^2)^2(ab)^{n-1}$
- $a^{n-1}bab^3a^{n-1}b^2a^nb^3a^nbab^2$
  $a^{n-1}b^2a^nbab^3a^nb^3a^{n-1}b^2ab$

Powers of Special Words Are Special

Suppose two words $w_1, w_2 \in \mathbb{F}_2$ are special. We prove by induction that powers of special words are special. First we treat the base cases, when the exponent n=2 and n=3.

Case 1: n=2

$$tr(w_1^2) = tr(w_1)tr(w_1) - tr(e) = tr(w_2)tr(w_2) - tr(e) = tr(w_2^2)$$

Case 2: n=3
Since, we proved $tr(w_1^2) = tr(w_2^2)$,
$tr(w_1^3) = tr(w_1 w_1)tr(w_1) - tr(w_1) = tr(w_2 w_2)tr(w_2) - tr(w_2) = tr(w_2^3)$ Now we suppose powers of special words are special for $n-1$, and prove it's true for n.

Powers of Special Words Are Special

Then n is either even or odd.

Case 1: $n = 2k + 1$ Since we know $n - 1 > k + 1$, for $n > 3$,
$tr(w_1^n) = tr(w_1^{k+1})tr(w_1^k) - tr(w_1) = tr(w_2^{k+1})tr(w_2^k) - tr(w_2) = tr(w_2^n)$.

Case 2: $n = 2k$. Since we know $n > k$,
$tr(w_1^n) = tr(w_1^k)tr(w_1^k) - tr(e) = tr(w_2^k)tr(w_2^k) - tr(e) = tr(w_2^n)$

Powers of Reverse Pairs Are Reverse

Proof.

Suppose two words that are reverse, $w_1, w_2 \in \mathbb{F}_2$ and $w_1 = \overleftarrow{w_2}$. Because of this, the powers

$$w_1^n, \ w_2^n$$

are equivalent to:

$$w_1^n, \ \overleftarrow{w_2}^n$$

Therefore powers of reverse pairs are reverse pairs. $\square$

All Special Words Have At Least 3 of Each Letter

Proof.

If two words have only one instance of a letter, then they will be cyclically equivalent. Since special words must have the same exponent, if they have two instances of a letter, they will either not have the same exponent or be cyclically equivalent. [Hor72] □

The Images of Non-Conjugate Words are Non-Conjugate

Proof.

Let ϕ be an automorphism and suppose the pair (w_1, w_2) is special. We attempt to prove by contradiction that $\phi(w_1)$ and $\phi(w_2)$ are not conjugate. So we suppose that $\phi(w_1)$ and $\phi(w_2)$ are conjugate. Then $\phi(w_1) = b\phi(w_2)b^{-1}$ for some b. Since ϕ is surjective, there exists some q, such that $\phi(q) = b$. Then $\phi(w_1) = \phi(q)\phi(w_2)\phi(q)^{-1} = \phi(qw_2q^{-1})$. Thus $w_1 = qw_2q^{-1}$ Since (w_1, w_2) are special, they cannot be conjugate. Thus there exists a contradiction, and $\phi(w_1)$ and $\phi(w_2)$ cannot be conjugate.

Now we prove their traces are equal. $\square$

The Images of Special Words Have Equal Traces

Proof.

Suppose w_1 and w_2 are special. Given that the word w_1 can be written in the format $a^{\alpha_1}b^{\beta_1}...a^{\alpha_n}b^{\beta_n}$. Taking the automorphism ϕ of the word yields us $\phi(a)^{\alpha_1}\phi(b)^{\beta_1}...\phi(a)^{\alpha_n}\phi(b)^{\beta_n}$. Since $\phi(a)$ and $\phi(b)$ are themselves words, using the word mapping, we can transform $\phi(a)$ and $\phi(b)$ into a product of $SL(2,\mathbb{C})$ matrices, when multiplied together is itself a matrix in $SL(2,\mathbb{C})$. The same result holds true when we apply ϕ to w_2. Thus $\phi(a)$ and $\phi(b)$ under the word mapping are two matrices in $SL(2,\mathbb{C})$. Thus the image of w_1 under ϕ and the word map can be represented as $C^{\alpha_1}D^{\beta_1}...C^{\alpha_n}D^{\beta_n}$, with $C = f(\phi(a))$ and $D = f(\phi(b))$. Since $a^{\alpha_1}b^{\beta_1}...a^{\alpha_n}b^{\beta_n}$ is special with w_2 for any two matrices in $SL(2,\mathbb{C})$, and since all we've done is change what matrix in $SL(2,\mathbb{C})$ is represented, $\phi(w_1)$ has the same trace $\phi(w_2)$. $\square$

Statement of a conjecture from Moira Chas

For each hyperbolic h metric on a pair of pants (the fundamental group of this surface is $\mathbb{F}_2$), we can associate to each word the minimal length of its geodesic representative. We call it $l_h : \mathbb{F}_2 \to \mathbb{R}^+$.

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It is conjectured that the repartition of those values among words of length N will , as N goes to infinity, follow a binomial law of variance $\sigma_h \times N$ and of mean value $\kappa_h \times N$. [CLM13]

Hyperbolic metric on a pair of pants

We have three distinguished elements in the fundamental group of a pair of pants, γ_1, γ_2 and γ_3 once their lengths are fixed, we get a unique metric on the pair of pants. Hence if we take γ_1 and γ_2 to be the generators of the free group, we just need to fix the lengths of γ_1, γ_2 and $\gamma_1\gamma_2$.

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We will use the following non-trivial fact to any metric we can associate a unique (up to conjugacy) triplet (x, y, z) given by the length of respectively $\gamma_1, \gamma_2, \gamma_3$. On the other hand, the monodromy operations shows that to each metric one can associate a unique conjugacy class of representations from $\mathbb{F}_2 \to SL_2\mathbb{R}$. This finally boils down by Fricke polynomial arguments to give the trace of the first generator according to this representation, the trace of the second one and the trace of the product of both.

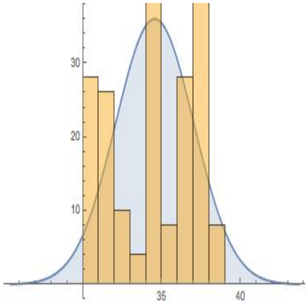
HYPERBOLIC METRIC ON A PAIR OF PANTS

Now from hyperbolic geometry, if we fix the metric h (and hence fix $x = tr(A)$, $y = tr(B)$ and $z = tr(AB)$) we can relate the geometric length $l_h(w)$ of an element w of $\mathbb{F}_2$ as the fundamental group of our pair of pants and the trace of w with respect to the representation, namely if P_w is the Fricke polynomial we have :

$$l_h(w) = 2arcosh(\frac{|P_w(x, y, z)|}{2})$$

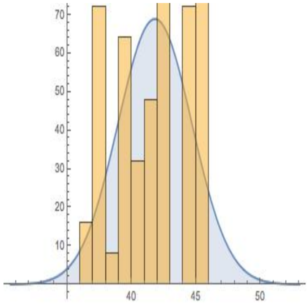
Does the population of trés special words fit the conjecture ? (test for $x = 3$, $y = 6$ and $z = 9$)

Figure: Distribution for length 20 with its normal distribution



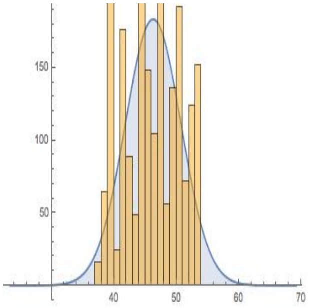
DOES THE POPULATION OF TRÉS SPECIAL WORDS FIT THE CONJECTURE ? (TEST FOR $x = 3$, $y = 6$ AND $z = 9$)

FIGURE: Distribution for length 23 with its normal distribution



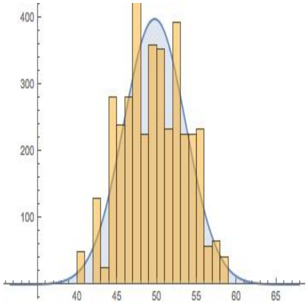
Does the population of trés special words fit the conjecture ? (test for $x = 3$, $y = 6$ and $z = 9$)

Figure: Distribution for length 26 with its normal distribution



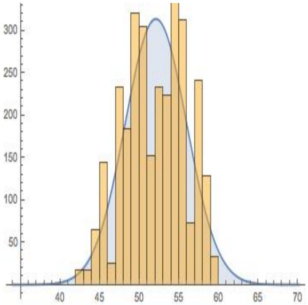
DOES THE POPULATION OF TRÉS SPECIAL WORDS FIT THE CONJECTURE ? (TEST FOR $x = 3$, $y = 6$ AND $z = 9$)

FIGURE: Distribution for length 27 with its normal distribution



Does the population of trés special words fit the conjecture ? (test for $x = 3$, $y = 6$ and $z = 9$)

Figure: Distribution for length 28 with its normal distribution



2-speciality and conjugacy classes

The question arises from the following remark, one can prove that if (w, w') is an inverse or reverse 2-special pair, for all $A, B \in SL_2\mathbb{C}$ we have that :
$w(A, B)$ and $w'(A, B)$ are conjugate to each other.

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Question : does this always hold for any 2-special pair ?

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Question : does this always hold for any 2-special pair ?

Using some characteristic polynomial argument if this does not hold then we may find a 2-special pair (w, w') and A, B such that one of the two following cases hold :

$$w(A, B) = I_2 \text{ and } w(A', B') \text{ is conjugate to } \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$w(A, B) = -I_2 \text{ and } w(A', B') \text{ is conjugate to } \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

Stop shooting in the dark with the patterns

Now that we have found many patterns, it would be a good idea to understand how they are related and then find a pattern for the patterns.

Most of the pattern we find follow this kind of rule, the special pair (w_n, w'_n) may be written as follow :

$$w_n = f_n u \text{ and } w'_n = f_n v$$

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Where f_n is a word whose length is increasing with n and u and v are fixed conjugate words or maybe reverse.

Go the other directions might be fruitful, namely give some conditions on (f_n) and (u, v) having the same traces such that $f_n u$ and $f_n v$ are 2-special with each other.

Proving non 3-speciality of infinite families of words

For the pattern we found, we proved 2-speciality for the whole family. However, we have just checked non 3-speciality computationally. Seeing the particular form of those words, it should not be too hard to prove that they are actually never 3-special pairs. This might be a warm up for the question of reverse pairs.

Study the action of $Out(\mathbb{F}_2)$ on its conjugacy classes

The property that $Out(\mathbb{F}_2)$ is fixing n-speciality is important, a study of the action of $Out(\mathbb{F}_2)$ on its conjugacy classes (minimal length within an orbit, is a 2-special pair in the class of a reverse pair ?) will undoubtedly help us to know which direction to take to find a 3-special word.

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